On Methods of Deducing Rates of Mortality and Withdrawal

WITH THEIR APPLICATION TO THE

EXPERIENCE AND VALUATION

OF

CLERKS' ASSOCIATIONS.

BY

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(I).

AN INVESTIGATION OF SOME OF THE METHODS FOR DEDUCING THE RATES OF MORTALITY, AND OF WITHDRAWAL, IN YEARS OF DURATION; WITH

(II).

THE APPLICATION OF SUCH METHODS TO THE COMPUTATION OF THE RATES EXPERIENCED, AND THE SPECIAL BENEFITS GRANTED, BY CLERKS' ASSOCIATIONS.

BY

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THE investigation of the experience, and valuation of the liabilities, of what are known as Clerks' Associations, present some features of special interest; and I have thought that it might be useful to discuss and explain some of the methods that have been employed in analyzing the data, and in deducing therefrom the tables of money values appropriate for the valuation of the risks.

In dealing with this subject I have had occasion to investigate the most suitable methods of deducing the numbers exposed to risk, and the rates of mortality and of withdrawal, as affected by the ages of the subscribing members, and by the duration of their membership respectively; and I propose, in the first part of the present paper, to discuss, in some detail, such of the methods suggested for dealing with the experience amongst assured lives,

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as may seem to attain, in whole or in part, the desired objects. I shall investigate the formulæ appropriate to give effect to these methods in a convenient tabular form; and shall illustrate their practical working, and compare their results, by an analysis of a portion of the experience of a Clerks' Association, the data of which I have recently investigated. I shall then seek to show the adaptability of the several methods and formulæ suggested to an investigation of lives insured in an assurance office, and the distinctive characteristics of such an experience; also, how the methods are modified in the case where the period of observation is limited by calendar years and by policy years respectively.

In the second portion of this paper, I propose to state the general characteristics of Clerks' Associations, as to contribution, benefit, and the like; and to investigate the methods of computing the values of their varying benefits at death; also of deducing the rate, and computing the value, of the special benefit during non-employment, granted by such associations; with due allowance throughout for the effect of withdrawals or secessions upon the values ascertained, and generally upon the valuation reserves.

(I). Investigation of Methods for deducing the Rates of Mortality and of Withdrawal in Years of Duration.

(A). Period of Observation limited by Calendar Years.*

Illustrative Experience.

It will be sufficient for the present to state (reserving further details for the second portion of this paper) that the data here employed for purposes of illustration formed part of the experience of a Clerks' Association during the period of five complete calendar years, 1888 to 1892 inclusive; that the age at entry recorded on the cards constituting the experience was in all cases the "office age" upon which the member's subscriptions at entry were based; that the subscriptions were throughout payable, at

* From some remarks made in the discussion which followed the reading of this essay, it seems necessary to point out that the limitation of the *period of observation* within complete calendar years is not to be confounded with the *tabulation and scheduling of the facts* according to the "calendar year" method. In both sections (A) and (B) of this paper the *facts* are tabulated according to "policy-years," or years of duration; but in section (A) the *observations* included are comprised within an integral number of calendar years, the period actually selected for illustration being from 1 January 1888, to 31 December 1892.

monthly intervals, upon the first day of each calendar month; and that the number of withdrawals or secessions was heavy, especially in the early years of duration.

With a view to investigating the rates of mortality and of withdrawal, as well as the rate of "non-employment", as affected by the age of the member, and by his duration, it was decided to tabulate the facts, and deduce the experience throughout, according to office ages at entry and years of duration. By this means, although at the expense of some additional labour, the material was readily available for ascertaining, in respect of each particular rate investigated, whether it was in the main a function of the age, or of the year of duration, or of both age and duration; and the material could be readily combined, in such a way as to give effect to the conclusions in this respect, as indicated by the experience.

The data available, in the particular case here dealt with, were, however, evidently not sufficiently extensive to permit of trustworthy results, as tabulated for each successive age at entry, and for each year of duration; and it was therefore determined to group the entry ages quinquennially, and to consider each group as representing the experience of the central age at entry in the group. Thus, cases entering at office ages 18, 19, 20, 21, and 22, were all classed together as entrants at "Central Age at Entry (20)", and so on; and, assuming that the numbers entering, and the rates obtaining, on either side of the central age, did not greatly differ, or so differed as to introduce compensating errors, there was evidently no material departure from the truth involved in this assumption.

The formulæ employed, and presently to be developed, are, however, applicable equally to the case of individual ages at entry; and where the experience is sufficiently extensive (as, for example, in the New Experience of the Institute of Actuaries), the natural and preferable course would be to tabulate the data, and deduce the results, for each separate age at entry.

The individual years of duration were, in all cases, scheduled and dealt with separately, and the duration being entered upon each card, the material was available for ascertaining, in each quinquennial group of entry-ages, and in each separate year of duration, the precise incidence of deaths, withdrawals, and of the benefit during non-employment, with a view to deducing the true rates of mortality, withdrawal, and benefit, respectively.

The several methods which have been proposed for dealing with a life experience according to "policy years", or years of duration, were all carefully referred to. These methods have been very usefully summarized and discussed by Mr. Ryan (*J.I.A.*, xxvi, 256), Mr. Chatham (*J.I.A.*, xxix, 81), and Mr. Whittall (*J.I.A.*, xxxi, 161); while some of them have been illustrated in a practical form by Dr. Sprague (*J.I.A.*, xxxi, 205) and Mr. Meikle (*J.I.A.*, xxxi, 229). None of the methods suggested, as applied by the above writers, appeared, however, in all respects to meet the case here investigated; and I have thought that it would be of interest to deal fully with their special application to this particular case, in the hope that a further discussion of these several methods might elucidate more fully their comparative advantages.

The methods which I have selected for illustration and comparison are—(1) the Exact Duration Method; (2) the Mean Duration Method; and (3) the Nearest Duration Method; each method being so applied as strictly to preserve the incidence of the cases throughout in their true years of duration.

I should have liked also to include in this investigation the special method suggested by Mr. G. F. Hardy and the late Mr. Rothery (*J.I.A.*, xxvii, 165), which may perhaps appropriately be described as the "Mean Age Method"; as well as that proposed by Mr. G. King (*J.I.A.*, xxvii, 218), which might be styled the "Nearest Age Method"; as also the somewhat similar, but in some respects inferior method, called by Dr. Sprague the "Final Age Method" (*J.I.A.*, xxxi, 215); and especially as these methods are all very simple and facile in their operations. It appeared to me, however, that, although these methods were doubtless admirably adapted for the purposes designed by their respective authors, they certainly do not tabulate the cases exposed to risk, and deduce the rates experienced, both of mortality and withdrawal, strictly in years of duration. This has been shewn, as it appears to me, clearly, by Mr. Whittall (*J.I.A.*, xxxi, 182-6); and I was, therefore, unable to include them as suitable for my present purpose. A further reason for their exclusion was, that they all involved data based upon the years of birth, or the birthdays, of the lives included; and, in the particular experience here selected for illustration of the selected methods, I had throughout no information whatever as to the dates of birth of the lives.

Exact Duration Method.

The durations of the cases were entered on the cards constituting the experience (1) as at the commencement of the period of observation, where the case was then in force; (2) as at the close of the period of observation, where the case was then existing; (3) as at exit during the period of observation, by death or by withdrawal. The duration was obtained in class (1) by taking the difference between the date of entry and January 1888; in class (2) by taking the difference between the date of entry and January 1893; and in class (3) by taking the difference between the date of entry and the date of exit. As all subscriptions were due on the first day of a calendar month, the durations were thus truly stated in integral months, excepting only in the cases of death, where the duration was stated with a possible error not exceeding a month (and averaging a fortnight) in any particular case. It may be added that this error would have been reduced by one-half, if the date of death had been entered on the cards, or computed for the purpose of ascertaining the duration, as the first day of the month *nearest* to the actual date of death*.

The durations were, however, recorded on the cards, not in years and months, but in years and equivalent decimals, correct to one decimal place. Thus, actual durations of 3 years and 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 months were entered as 3·1, 3·2, 3·3, 3·3, 3·4, 3·5, 3·6, 3·7, 3·7, 3·8, 3·9. The fractional durations, thus expressed decimally, were found to be much more convenient, in practical use, than if expressed in years and months, being more easily cast and aggregated by inspection, and adapting themselves much more readily to the subsequent processes. Upon the assumption, which was found to be fully justified in this particular experience, that members are equally likely to enter under observation, or to withdraw, in any month of the year, the decimal expressions will be found to give an average result in which the balance of errors is equal; the aggregate of the decimal expressions for the successive months of duration

$$(0+\cdot1+\cdot2+\cdot3+\cdot3+\cdot4+\cdot5+\cdot6+\cdot7+\cdot7+\cdot8+\cdot9),$$

and of the actual months of duration

$$(0+1+2+3+4+5+6+7+8+9+10+11),$$

being both equal to 5·5 years or 66 months.

* If this modification be introduced, care must, however, be taken that the cases of death are throughout located in their true years of duration.

While this plan was found to work well, and to introduce no perceptible error, in the particular experience here investigated, it is probable that, in the case of ordinary policy investigations, where the assurances will have a strong tendency to terminate at or near to quarterly intervals in the year of duration, the current quarter at exit, expressed decimally as ·25, ·50, ·75, or 1·00, would give, on the whole, more satisfactory results.

The durations having thus been recorded upon the cards, they were first sorted into quinquennial groups of ages at entry, all cases entering at office ages 18 to 22 inclusive being grouped together as "Central Age at Entry (20)"; entrants between ages 23 to 27 inclusive as "Central Age (25)"; and so on, up to central age (45), which included the highest age at entry admissible under the rules. There were thus six groups of central ages at entry.

Taking now the group of cases entering at office ages 18 to 22 inclusive, or at the central age (20), which group I shall throughout employ to illustrate the methods here followed, the cards constituting the group were sorted into "survivors," in force on 1 January 1888, and "new entrants" coming on the books during the five years 1888-92 inclusive. The new entrants were counted, and their total number recorded. The "survivors" were then sorted, according to the curtate duration at entry on observation, as recorded on the cards, and so that all cases, for example, whose recorded duration at the commencement of the period ranged from 5·1 to 6·0 inclusive, were grouped together as of "curtate" duration 5. The number of survivors in each year of duration was then counted, and recorded in a tabular form; and the aggregate fractional exposure of the cases in each year of duration as recorded on the cards was cast by inspection, and tabulated against the number of corresponding cases.

The operation of casting the decimal figures, disregarding the integers, is rapid and easy; the only point to note being that, according to the method of classification here suggested, an exact duration of 6·0 (for instance) is considered as of "curtate" duration 5 and "fractional" exposure 1·0; but this will be found in practice to give rise to no difficulty whatever.

It may be added that, in respect of the survivors at the commencement, as well as of the cases existing at the close, of the observation, it seems to be quite immaterial whether an

integral duration of (say) 6·0 is classified among cases having six years duration and upwards (and counted as 0) or among cases having five years' duration and upwards (and counted as 1). In the case, however, of withdrawals and deaths, it seems to be clear that they can only be properly treated by considering them as cases of withdrawal (or of death) occurring at the end of the sixth year of membership, and therefore classifying them among emergents having five years' duration and upwards. This will be evident if consideration be given to the withdrawals, for instance, at the end of twelve months' duration, which should clearly, as it appears to me, be tabulated with the withdrawals occurring in the first year, for the purpose of correctly deducing the rate of withdrawal in that year.

The sorting would have been more symmetrical and a little more facile in this respect if all cases of fractional duration had been entered upon the cards at the next higher integer, or what may be called the "current year of duration" (leaving, of course, integral durations unaltered); and in this case the fractional period would have had to be separately dealt with by way of deduction. This appears to have been the plan most frequently followed in published investigations; but, after careful consideration, I arrived at the conclusion that the disadvantages of this plan of deduction would on the whole be greater than the slight difficulty involved in the special treatment of the cases of integral duration. To provide against any risk of possible error in the tabulations, a sort of "danger-signal" may be put up, by underlining the ·0 in red ink upon the cards, in all cases of integral duration, as a reminder to the clerk that they are each to be counted for a full unit.

The number of original entrants, and the number and aggregate fractional duration at entry, of the "survivors" in each year of duration, having been thus recorded, the cards were combined, and re-sorted into cases *existing* at the close of the period of observation; cases *withdrawing* during the period; and cases *dying* during the period. The cards in each of these three groups were then sorted according to their "curtate" durations (for the existing) at the close of the observation, or (for withdrawals and deaths) at exit; and the number of cases, and their aggregate fractional exposures, were then tabulated in their appropriate years of duration.

The following Schedule (A) shews the form in which the

cases entering at "central age at entry (20)", and their fractional durations at entry and at exit, were tabulated, as a result of the above processes of sorting and grouping.

It will be remarked that the aggregate fractional exposures, as deduced from the cards, represents in the case of the "survivors" that portion of the current year of duration already expired at the commencement of the period of observation; and in the case of the "existing," "withdrawals," and "deaths," that portion of the current year of duration over which the cases were actually at risk during the period of observation.

It will also be observed that the fractional exposure of the death cases, as well as of the withdrawals, is, in this investigation, terminated at the date of exit, whether by death or withdrawal, and not continued until the end of the year of duration current at death. This was done advisedly, as the numbers exposed to risk, so arrived at, formed at the same time a suitable denominator for computing the rate of allowance during unemployment, and a convenient basis for deducing, by a simple modification, the denominators appropriate for the calculation of the rates of mortality, and of withdrawal, respectively. The formulæ adopted were also, throughout, more symmetrical and convenient upon this basis.

I now proceed to state the formulæ for deducing the number exposed to risk, and the rates of mortality and withdrawal. A definition of the leading symbols employed throughout this paper may first be given.

Let $\overline{E}_{[x]+t}$= the number exposed to risk, in respect of cases entering at "office age" x, during the $(t+1)$th year of duration; the bar over the E indicating that the number exposed is computed up to the actual cessation of the risk, whether by death, withdrawal, or close of the period of observation;* and let

$E_{[x]+t}$= the number exposed to the risk of death

and $(wE)_{[x]+t}$= the number exposed to the risk of withdrawal during the $(t+1)$th year of duration; where the cases of death and of withdrawal are respectively given a full year's exposure in the year of duration current at exit.

*The function $\overline{E}_{[x]+t}$ really represents the number exposed [in the $(t+1)$th year] *to risk of death or withdrawal;* and would be appropriately employed in calculations (1) of benefits the continuance of which depends upon the member's being alive and in full membership (such as the annuity by which the members' subscriptions would be valued) (2) of benefits which would necessarily cease on the occurrence of either death or withdrawal (such as an allowance during sickness or non-employment).

Schedule (A).—Central Age at Entry (20).

Table of the Numbers Surviving, Existing, Withdrawing, and Dying, in Years of Duration; with Fractional Exposure of the Survivors, as at Entry on Observation, and of the Existing, Withdrawals and Deaths, as at Close of Observation, or Exit.

Curtate Duration*	Survivors		Existing		Withdrawals		Deaths		Curtate Duration*
	Cases	Fractional Exposure	Cases	Fractional Exposure	Cases	Fractional Exposure	Cases	Fractional Exposure	
t	$s_{[x]+t}$	$s'_{[x]+t}$	$e_{[x]+t}$	$e'_{[x]+t}$	$w_{[x]+t}$	$w'_{[x]+t}$	$d_{[x]+t}$	$d'_{[x]+t}$	t
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	137	81·8	82	44·2	51	24·4	1	0·5	0
1	101	56·5	83	43·3	47	27·1	5	3·6	1
2	132	73·2	7	6·1	44	25·5	3	1·3	2
3	120	65·8	72	40·1	45	20·2	1	0·5	3
4	129	73·9	75	43·1	55	25·3	3	1·3	4
5	101	56·3	69	41·1	44	25·3	3	1·1	5
6	77	37·8	62	37·9	33	18·7	...	...	6
7	89	53·6	74	43·4	23	13·5	4	2·3	7
8	87	49·2	82	45·5	27	16·0	5	2·8	8
9	76	44·2	89	51·1	16	7·9	1	0·3	9
10	64	37·7	69	39·6	19	8·8	3	1·8	10
11	48	24·0	48	24·2	20	10·2	6	3·3	11
12	15	7·6	65	39·9	11	6·6	2	1·0	12
13	19	11·1	69	37·7	13	6·5	2	0·3	13
14	19	11·0	59	34·7	11	6·0	1	0·6	14
15	10	7·2	46	28·0	2	0·7	2	1·1	15
16	13	7·7	35	17·7	5	3·2	...	...	16
17	15	9·6	11	6·5	3	2·3	...	...	17
18	10	3·8	13	6·5	3	1·1	...	...	18
19	4	3·0	16	9·7	1	0·8	...	...	19
20	9	7·2	7	5·1	3	1·8	...	...	20
21	10	6·6	11	7·1	1	0·6	1	0·2	21
22	15	8·1	10	6·0	...	...	...	...	22
23	16	6·6	9	3·5	1	0·2	...	...	23
24	10	6·5	4	3·0	...	...	...	...	24
25	7	3·0	9	7·2	1	0·1	1	0·6	25
26	...	...	10	6·6	1	0·8	1	0·5	26
27	...	...	12	5·6	...	...	1	0·9	27
28	...	...	14	5·6	1	0·9	...	...	28
29	...	...	8	5·8	...	...	2	0·3	29
30	...	...	5	2·6	...	...	...	...	30
	1,333	753·0	1,225	698·4	481	254·5	48	24·3	

* Integral durations of $(t+1)$ years being treated throughout as of "curtate" duration (t), and "fractional exposure" (1·0).

Note.—The above experience represents a small portion only of the available data, the cases entering at the grouped entry ages 18 to 22 inclusive having been selected solely for the purpose of illustrating the different methods employed.

Also, let $q_{[x]+t}$ be the annual rate of mortality; and
$(wq)_{[x]+t}$ the annual rate of withdrawal, in the $(t+1)$th year of duration.

Let $s_{[x]+t}=$ the *survivors* in force at the commencement of the period of observation, having a duration exceeding t, but not exceeding $(t+1)$ years;

$n_{[x]}=$ the *new entrants* at "office age" x, during the period of observation;

$e_{[x]+t}=$ the cases *existing* at the close of the period of observation, having a duration exceeding t, but not exceeding $(t+1)$ years;

$w_{[x]+t}=$ the *withdrawals* during the period of observation, having a duration at exit exceeding t, but not exceeding $(t+1)$ years;

$d_{[x]+t}=$ the *deaths* during the period of observation, having a duration at death exceeding t, but not exceeding $(t+1)$ years.

Also, let $s'_{[x]+t}=$ the exact fractional exposure of the $s_{[x]+t}$ survivors, computed from the commencement of the $(t+1)$th year of duration, up to the commencement of the period of observation;

$e'_{[x]+t}=$ the exact fractional exposure of the $e_{[x]+t}$ cases existing, computed from the commencement of $(t+1)$th year of duration, up to the close of the period of observation;

$w'_{[x]+t}=$ the exact fractional exposure of the $w_{[x]+t}$ cases of withdrawal, computed from the commencement of the $(t+1)$th year of duration, up to the date of withdrawal;

$d'_{[x]+t}=$ the actual fractional exposure of the $d_{[x]+t}$ cases of death, computed from the commencement of the $(t+1)$th year of duration, up to the date of death;

Also, let $$(e+w+d)=f,$$

that is, the *total decrement* in the year of duration, in respect of cases existing, withdrawing, and dying; and let

$$(s-f)=g,$$

that is, the *net movement* in the year of duration, among cases surviving, existing, withdrawing, and dying; and similarly, let

$$(e'+w'+d')=f'$$
$$(s'-f')=g'$$

where f' and g' represent the aggregate fractional exposures of the f and g cases respectively.

Then we have (*see* Appendix A):

$$\overline{\mathrm{E}}_{[x]}=n_{[x]}+g_{[x]}-g'_{[x]} \quad \ldots\ldots\ldots \quad (1)$$

$$\overline{\mathrm{E}}_{x]+t}=\overline{\mathrm{E}}_{[x]+t-1}+g_{[x]+t}-\Delta\, g'_{[x]+t-1} \quad \ldots \quad (3)^*$$

formulæ by which the numbers exposed to risk in the first year, and those in successive years, can be continuously computed.

Also, $$\overline{\mathrm{E}}_{[x]+t}=n_{[x]}+\Sigma_0^t(g)-g'_{[x]+t} \quad \ldots\ldots \quad (4)$$

an alternative formula for deducing the numbers exposed to risk by summation, in a convenient tabular form.

The appended Schedule (B) shows the arrangement of the data, and the computations for deducing, in respect of the group of entrants at "central age at entry (20)", the numbers exposed to risk, and the rates experienced, in successive years of duration. The cases surviving, existing, withdrawing, and dying, are entered respectively in columns (2), (3), (4) and (5); and under them are placed the corresponding fractional exposures of the cases, printed in a distinctive type. Column (6) gives the value of the total decrement $f(=e+w+d)$, and the corresponding fractional exposures $f'(=e'+w'+d')$; and column (7) gives the value of the net movement of cases $g(=s-f)$. In columns (8) to (10) the numbers exposed to risk are deduced by the summation formula (4); the values of (g) being continuously summed and added to the number of original entrants (n), in column (8); while the value of $g'(=s'-f')$ entered in column (9) is deducted from the values in column (8), and the result, entered in column (10), gives the value of the number exposed to risk $\overline{\mathrm{E}}_{[x]+t}$. In columns (11) to (13) the alternative formula (3) is employed, the value of $\Delta g'_{[x]+t-1}$ being entered in column (11) and deducted from the values of $g_{[x]+t}$ in column (7), and the difference entered in column (12); the numbers exposed to risk being then obtained by continuous addition in column (13), starting with the value of $n_{[x]}$. In column (14) the numbers exposed to the risk of death are obtained by the formula

$$\mathrm{E}_{[x]+t}=\overline{\mathrm{E}}_{[x]+t}+(d-d')_{[x]+t},$$

and the numbers exposed to the risk of withdrawal in column (16) by the formula

$$(wE)_{[x]+t}=\overline{\mathrm{E}}_{[x]+t}+(w-w')_{[x]+t} \text{ (\textit{see} Appendix C).}$$

* The formulæ are numbered consecutively in the Appendices; and to avoid confusion, the same numbering has been employed in those formulæ cited in the text.

Schedule (B).—Observation Extending over

Table showing alternative methods of deducing the Numbers Exposed to … duration, and with exact Fractional

Curtate Duration	Survivors	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	Summation Method		
t	$s_{[x]+t}$ $s'_{[x]+t}$	$e_{[x]+t}$ $e'_{[x]+t}$	$w_{[x]+t}$ $w'_{[x]+t}$	$d_{[x]+t}$ $d'_{[x]+t}$	$f_{[x]+t}$ $f'_{[x]+t}$	$g_{[x]+t}$ $=(s-f)$	$n_{[x]}$ $+\Sigma_0^t(g)$	$g'_{[x]+t}$ $=(s'-f')$	$E_{[x]+t}$ $=(8)-(9)$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
							$n_{[x]}=421$		
0	137	82	51	1	134	+ 3	424		411·3
	81·8	44·2	24·4	0·5	69·1			+ 12·7	
1	101	83	47	5	135	− 34	390		407·5
	56·5	43·3	27·1	3·6	74·0			− 17·5	
2	132	7	44	3	54	+ 78	468		427·7
	73·2	6·1	25·5	1·3	32·9			+ 40·3	
3	120	72	45	1	118	+ 2	470		465·0
	65·8	40·1	20·2	0·5	60·8			+ 5·0	
4	129	75	55	3	133	− 4	466		461·8
	73·9	43·1	25·3	1·3	69·7			+ 4·2	
5	101	69	44	3	116	− 15	451		462·2
	56·3	41·1	25·3	1·1	67·5			− 11·2	
6	77	62	33	..	95	− 18	433		451·8
	37·8	37·9	18·7	..	56·6			− 18·8	
7	89	74	23	4	101	− 12	421		426·6
	53·6	43·4	13·5	2·3	59·2			− 5·6	
8	87	82	27	5	114	− 27	394		409·1
	49·2	45·5	16·0	2·8	64·3			− 15·1	
9	76	89	16	1	106	− 30	364		379·1
	44·2	51·1	7·9	0·3	59·3			− 15·1	
10	64	69	19	3	91	− 27	337		349·5
	37·7	39·6	8·8	1·8	50·2			− 12·5	
11	48	48	20	6	74	− 26	311		324·7
	24·0	24·2	10·2	3·3	37·7			− 13·7	
12	15	65	11	2	78	− 63	248		287·9
	7·6	39·9	6·6	1·0	47·5			− 39·9	
13	19	69	13	2	84	− 65	183		216·4
	11·1	37·7	6·5	0·3	44·5			− 33·4	
14	19	59	11	1	71	− 52	131		161·3
	11·0	34·7	6·0	0·6	41·3			− 30·3	
15	10	46	2	2	50	− 40	91		113·6
	7·2	28·0	0·7	1·1	29·8			− 22·6	
16	13	35	5	..	40	− 27	64		77·2
	7·7	17·7	3·2	..	20·9			− 13·2	
17	15	11	3	..	14	+ 1	65		64·2
	9·6	6·5	2·3	..	8·8			+ 0·8	
18	10	13	3	..	16	− 6	59		62·8
	3·8	6·5	1·1	..	7·6			− 3·8	
19	4	16	1	..	17	− 13	46		53·5
	3·0	9·7	0·8	..	10·5			− 7·5	
20	9	7	3	..	10	− 1	45		44·7
	7·2	5·1	1·8	..	6·9			+ 0·3	
21	10	11	1	1	13	− 3	42		43·3
	6·6	7·1	0·6	0·2	7·9			− 1·3	
22	15	10	..	..	10	+ 5	47		44·9
	8·1	6·0	..	..	6·0			+ 2·1	
23	16	9	1	..	10	+ 6	53		50·1
	6·6	3·5	0·2	..	3·7			+ 2·9	
24	10	4	..	..	4	+ 6	59		55·5
	6·5	3·0	..	..	3·0			+ 3·5	
25	7	9	1	1	11	− 4	55		59·9
	3·0	7·2	0·1	0·6	7·9			− 4·9	
26	..	10	1	1	12	− 12	43		50·9
	..	6·6	0·8	0·5	7·9			− 7·9	
27	..	12	..	1	13	− 13	30		36·5
	..	5·6	..	0·9	6·5			− 6·5	
28	..	14	1	..	15	− 15	15		21·5
	..	5·6	0·9	..	6·5			− 6·5	
29	..	8	..	2	10	− 10	5		11·1
	..	5·8	..	0·3	6·1			− 6·1	
30	..	5	..	..	5	− 5	..		2·6
	..	2·6	..	..	2·6			− 2·6	
	1,333	1,225	481	48	1,754	− 421	6,210	..	6,434·2
	753·0	698·4	234·5	24·3	977·2	..	..	− 224·2	..

FIVE CALENDAR YEARS.—EXACT DURATION METHOD.—SCHEDULE (B).
Risk, and the Rates, of Mortality and of Withdrawal, in true years of Exposures.—Central Age at Entry (20).

Continuous Method			Mortality		Withdrawal		Curtate Duration
			Exposed	Rate	Exposed	Rate	
$\Delta g'_{[x]+t-1}$	$g_{[x]+t} - \Delta g'_{[x]+t-1}$	$\overline{E}_{[x]+t}$	$\overline{E}_{[x]+t} = \overline{E} + (d - d')$	$q_{[x]+t} = \frac{d}{\overline{E}}$	$(wE)_{[x]+t} = \overline{E} + (w - w')$	$(wq)_{[x]+t} = \frac{w}{wE}$	t
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
		$n_{[x]} = 421$					
		411·3	411·8	·00243	437·9	·1165	0
+ 12·7	− 9·7						
		407·5	408·9	·01223	427·4	·1100	1
− 30·2	− 3·8						
		427·7	429·4	·00699	446·2	·0986	2
+ 57·8	+ 20·2						
		465·0	465·5	·00215	489·8	·0919	3
− 35·3	+ 37·3						
		461·8	463·5	·00647	491·5	·1119	4
− 0·8	− 3·2						
		462·2	464·1	·00646	480·9	·0915	5
− 15·4	+ 0·4						
		451·8	451·8	..	466·1	·0708	6
− 7·6	− 10·4						
		426·6	428·3	·00934	436·1	·0527	7
+ 13·2	− 25·2						
		409·1	411·3	·01216	420·1	·0643	8
− 9·5	− 17·5						
		379·1	379·8	·00263	387·2	·0413	9
0	− 30·0						
		349·5	350·7	·00856	359·7	·0528	10
+ 2·6	− 29·6						
		324·7	327·4	·01833	334·5	·0598	11
− 1·2	− 24·8						
		287·9	288·9	·00692	292·3	·0376	12
− 26·2	− 36·8						
		216·4	218·1	·00917	222·9	·0583	13
+ 6·5	− 71·5						
		161·3	161·7	·00618	166·3	·0662	14
+ 3·1	− 55·1						
		113·6	114·5	·01747	114·9	·0174	15
+ 7·7	− 47·7						
		77·2	77·2	..	79·0	·0633	16
+ 9·4	− 36·4						
		64·2	64·2	..	64·9	·0462	17
+ 14·0	− 13·0						
		62·8	62·8	..	64·7	·0464	18
− 4·6	− 1·4						
		53·5	53·5	..	53·7	·0186	19
− 3·7	− 9·3						
		44·7	44·7	..	45·9	·0654	20
+ 7·8	− 8·8						
		43·3	44·1	·02268	43·7	·0229	21
− 1·6	− 1·4						
		44·9	44·9	..	44·9	..	22
+ 3·4	+ 1·6						
		50·1	50·1	..	50·9	·0197	23
+ 0·8	+ 5·2						
		55·5	55·5	..	55·5	..	24
+ 0·6	+ 5·4						
		59·9	60·3	·01658	60·8	·0165	25
− 8·4	+ 4·4						
		50·9	51·4	·01945	51·1	·0196	26
− 3·0	− 9·0						
		36·5	36·6	·02732	36·5	..	27
+ 1·4	− 14·4						
		21·5	21·5	..	21·6	·0463	28
0	− 15·0						
		11·1	12·8	·15625	11·1	..	29
+ 0·4	− 10·4						
		2·6	2·6	..	2·6	..	30
+ 3·5	− 8·5						
..	..	6,434·2	6,457·9	..	6,660·7	..	
− 2·6	− 418·4	..	..	..	..		

Finally, in columns (15) and (17) the rates of mortality and of withdrawal are deduced by the respective formulæ

$$q_{[x]+t}=\frac{d_{[x]+t}}{E_{[x]+t}}$$

and

$$(wq)_{[x]+t}=\frac{w_{[x]+t}}{(wE)_{[x]+t}}$$

In practice, one or other of the two methods set out in columns (8) to (10), and (11) to (13), would alone be adopted, thus reducing the labour and the number of columns involved; and it will probably be found preferable to adopt the continuous formula shown in columns (11) to (13), and to employ the summation formula in verification of the results at any stage, thus

$$\begin{aligned}\bar{E}_{[20]+10}&=n_{[20]}+\Sigma_0^{10}(g)-g'_{[20]+10}\\&=421-84+12{\cdot}5\\&=349{\cdot}5\end{aligned}$$

$$\begin{aligned}\bar{E}_{[20]+20}&=n_{[20]}+\Sigma_0^{20}(g)-g'_{[20]+20}\\&=421-376-0{\cdot}3\\&=44{\cdot}7\end{aligned}$$

$$\begin{aligned}\bar{E}_{[20]+30}&=n_{[20]}+\Sigma_0^{30}(g)-g'_{[20]+30}\\&=421-421+2{\cdot}6\\&=2{\cdot}6\end{aligned}$$

I have, however, preferred to set out the full process under both methods, partly to illustrate the two operations, and partly to secure verification throughout.

The rates of mortality as set out in column (15), being based upon only 48 deaths, are, of course, very irregular, and I need hardly say that they are only computed here for purposes of illustration and comparison, and are not presented as representing results practically available. The withdrawals are 481 in number, and the rates of withdrawal, as set out in column (17), show a smoother progression in years of duration.

Bearing in mind that the Exact Duration Method gives effect to the precise exposures of the cases, and deduces the rates, both of mortality and of withdrawal, strictly as experienced in each successive year of duration, it may, I think, be considered as not unduly laborious for the valuable results obtained. Under this method no assumptions whatever are made, either as regards the ages attained, or as to the average epochs of entry or of exit, in the several years of duration.

In comparing the extent of the Schedule (B) with other tabular statements which have been published, it must be borne in mind that in the appended schedule the numbers exposed to risk and the rates experienced are deduced both for mortality and withdrawal.

Mean Duration Method.

If we consider that upon the average the fractional exposure of the survivors, in the years of duration current at the commencement of the period of observation, and of the cases existing, withdrawing, and dying, in the years of duration current at the close of the period, or at exit, is approximately equal to half a year, formulæ (1) and (3) will become [*see* Appendix (A)]

$$\overline{E}_{[x]}=n_{[x]}+\frac{g_{[x]}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\overline{E}_{[x]+t}=\overline{E}_{[x]+t-1}+\tfrac{1}{2}(g_{[x]+t-1}+g_{[x]+t}) \quad . \quad . \quad . \quad (6)$$

a very simple and convenient formula for computing the numbers exposed to risk by a continuous operation.

Similarly, formula (4) will become, upon the above assumptions,

$$\overline{E}_{[x]+t}=n_{[x]}+\Sigma_0^t(g)-\tfrac{1}{2}(g_{[x]+t}) \quad . \quad . \quad . \quad . \quad (7)$$

a formula by which the number exposed to risk can be obtained by tabular summation, or which can be applied as a useful check upon the results obtained by formula (6).

In this case it is not, of course, necessary to record the exact duration of the cases upon the cards, but only, in the case of the "survivors", the "curtate" duration at entry; and in the case of the existing, withdrawing, and dying, the "curtate" duration at the close of the observation, or at exit; cases of integral duration of $(t+1)$ years being treated, as before, as of "curtate" duration (t). The cards of the "survivors" must then be sorted and tabulated according to the duration at entry, and those of the cases existing, withdrawing, and dying, according to the duration at exit, as recorded upon the cards.

The appended Schedule (C) shows the process followed in computing the number exposed to risk $(\overline{E})$ by the two formulæ (6) and (7), also the values of E and (wE) deduced from the formulæ

$$E_{[x]+t}=\overline{E}_{[x]+t}+\frac{d}{2}$$

$$\text{and } (wE)_{[x]+t}=\overline{E}_{[x]+t}+\frac{w}{2} \qquad (\textit{See } \text{Appendix C}).$$

SCHEDULE (C).—OBSERVATION EXTENDING OVER

Table showing alternative methods of deducing the Numbers Exposed to duration, and with mean or average Fractional

Curtate Duration	Survivors	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	Summation Method		
t	$s_{[x]+t}$	$e_{[x]+t}$	$w_{[x]+t}$	$d_{[x]+t}$	$f_{[x]+t}$	$g_{[x]+t} = (s-f)$	$n_{[x]} + \Sigma_0^t(g)$	$-\frac{g_{[x]+t}}{2}$	$\bar{E}_{[x]+t}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8) $n_{[x]}=421$	(9)	(10)
0	137	82	51	1	134	+ 3	424	− 1·5	422·5
1	101	83	47	5	135	−34	390	+17·0	407·0
2	132	7	44	3	54	+78	468	−39·0	429·0
3	120	72	45	1	118	+ 2	470	− 1·0	469·0
4	129	75	55	3	133	− 4	466	+ 2·0	468·0
5	101	69	44	3	116	−15	451	+ 7·5	458·5
6	77	62	33	...	95	−18	433	+ 9·0	442·0
7	89	74	23	4	101	−12	421	+ 6·0	427·0
8	87	82	27	5	114	−27	394	+13·5	407·5
9	76	89	16	1	106	−30	364	+15·0	379·0
10	64	69	19	3	91	−27	337	+13·5	350·5
11	48	48	20	6	74	−26	311	+13·0	324·0
12	15	65	11	2	78	−63	248	+31·5	279·5
13	19	69	13	2	84	−65	183	+32·5	215·5
14	19	59	11	1	71	−52	131	+26·0	157·0
15	10	46	2	2	50	−40	91	+20·0	111·0
16	13	35	5	...	40	−27	64	+13·5	77·5
17	15	11	3	...	14	+ 1	65	− 0·5	64·5
18	10	13	3	...	16	− 6	59	+ 3·0	62·0
19	4	16	1	...	17	−13	46	+ 6·5	52·5
20	9	7	3	...	10	− 1	45	+ 0·5	45·5
21	10	11	1	1	13	− 3	42	+ 1·5	43·5
22	15	10	...	...	10	+ 5	47	− 2·5	44·5
23	16	9	1	...	10	+ 6	53	− 3·0	50·0
24	10	4	...	...	4	+ 6	59	− 3·0	56·0
25	7	9	1	1	11	− 4	55	+ 2·0	57·0
26	...	10	1	1	12	−12	43	+ 6·0	49·0
27	...	12	...	1	13	−13	30	+ 6·5	36·5
28	...	14	1	...	15	−15	15	+ 7·5	22·5
29	...	8	...	2	10	−10	5	+ 5·0	10·0
30	...	5	...	...	5	− 5	...	+ 2·5	2·5
	1,333	1,225	481	48	1,754	−421	6,210	210·5	6420·5

The values of $q_{[x]+t}$ and $(wq)_{[x]+t}$, as deduced from these numbers exposed to risk, are also appended. Here, again, the columns (8) to (10) can be dispensed with, and the summation formula employed simply for purposes of verification at suitable intervals.

This method has the advantages of being very simple and rapid in working, of avoiding, to a great extent, the employment of fractions, and, at the same time, of preserving the incidence of the cases strictly in their appropriate years of duration. It treats all cases of entry, of emergence, and of existence, as occurring in the middle of the year of duration: and one effect of this is that "surviving" cases entering and emerging in the same year of

Five Calendar Years.—Mean Duration Method.—Schedule (C).

Risk, and the Rates, of Mortality and of Withdrawal, in true years of Exposures.—Central Age at Entry (20).

Continuous Method		Mortality		Withdrawal		Curtate Duration
		Exposed	Rate	Exposed	Rate	
$\frac{g_{[x]+t-1}+g_{[x]+t}}{2}$	$\overline{E}_{[x]+t}$	$E_{[x]+t} = \overline{E}+\frac{d}{2}$	$q_{[x]+t} = \frac{d}{\overline{E}}$	$(wE)_{[x]+t} = \overline{E}+\frac{w}{2}$	$(wq)_{[x]+t} = \frac{w}{wE}$	t
(11)	(12)	(13)	(14)	(15)	(16)	(17)
	$n_{[x]}=421$					
+ 1·5	422·5	423·0	·00237	448·0	·1138	0
− 15·5	407·0	409·5	·01221	430·5	·1092	1
+ 22·0	429·0	430·5	·00697	451·0	·0982	2
+ 40·0	469·0	469·5	·00213	491·5	·0917	3
− 1·0	468·0	469·5	·00639	495·5	·1110	4
− 9·5	458·5	460·0	·00652	480·5	·0915	5
− 16·5	442·0	442·0	...	458·5	·0719	6
− 15·0	427·0	429·0	·00932	438·5	·0524	7
− 19·5	407·5	410·0	·01220	421·0	·0641	8
− 28·5	379·0	379·5	·00264	387·0	·0413	9
− 28·5	350·5	352·0	·00852	360·0	·0528	10
− 26·5	324·0	327·0	·01835	334·0	·0599	11
− 44·5	279·5	280·5	·00713	285·0	·0386	12
− 64·0	215·5	216·5	·00924	222·0	·0586	13
− 58·5	157·0	157·5	·00635	162·5	·0679	14
− 46·0	111·0	112·0	·01786	112·0	·0179	15
− 33·5	77·5	77·5	...	80·0	·0625	16
− 13·0	64·5	64·5	...	66·0	·0455	17
− 2·5	62·0	62·0	...	63·5	·0472	18
− 9·5	52·5	52·5	...	53·0	·0189	19
− 7·0	45·5	45·5	...	47·0	·0638	20
− 2·0	43·5	44·0	·02273	44·0	·0227	21
+ 1·0	44·5	44·5	...	44·5	...	22
+ 5·5	50·0	50·0	...	50·5	·0198	23
+ 6·0	56·0	56·0	...	56·0	...	24
+ 1·0	57·0	57·5	·01739	57·5	·0174	25
− 8·0	49·0	49·5	·02020	49·5	·0202	26
− 12·5	36·5	37·0	·02703	36·5	...	27
− 14·0	22·5	22·5	...	23·0	·0435	28
− 12·5	10·0	11·0	·18182	10·0	...	29
− 7·5	2·5	2·5	...	2·5	...	30
− 2·5						
−421	6,420·5	6,444·5	...	6,661·0	...	

duration are altogether eliminated from the experience. Thus, a case entering upon the period of observation at a duration of 2·1 years, and emerging at a duration of 2·9 years, is considered as entering upon observation at 2·5 years, and emerging at 2·5 years. As some compensation for this, a case entering (for instance) at 2·9 years and emerging at 3·1 years is considered as entering at 2·5 and emerging at 3·5, and as under observation for a full year. Upon the whole, however, the method deals with the fractional exposures fairly at average values; and in the case of an experience such as that here investigated, gives, as will be seen, values for the numbers exposed to risk, and for the rates of

mortality and of withdrawal, in successive years of duration, which agree very closely with those deduced by the Exact Duration Method.

Nearest Duration Method.

This is the method which has been illustrated by Dr. Sprague (*J.I.A.*, xxxi, 208–12) in schedules arranged according to ages at entry and years of duration. It has, I think, been somewhat hastily assumed that this method necessarily involves a "mixing-up of the policy years" (*J.I.A.*, xxxi, 309–315), and that, therefore, although possessing manifest and acknowledged advantages by way of simplicity of arrangement and facility of computation, it is unsuitable for an investigation (such as the present) which aims at deducing the true experience of each year of duration. It is mainly with a view to a further discussion of this question that I here introduce this method; and I shall illustrate its application by the same partial experience which has served to illustrate the Exact Duration Method and the Mean Duration Method.

The Nearest Duration Method proceeds upon the assumption that all cases entering on the period of observation do so either at the commencement or at the end of the year of duration current at entry; that all cases emerging during the period do so either at the commencement or at the end of the year of duration current at exit; and, consequently, that all cases existing at the close of the period have then completed integral years of duration. These assumptions involve the reference of all cases of fractional exposure to the beginning or to the end of the year of duration current at entry, or at exit; and in carrying this into effect, the *nearest* boundary of the year of duration is in all cases adopted. Thus, cases having a duration at entry (or at exit) of 6·1 to 6·4 years inclusive, would be considered as entering (or emerging) at an integral duration of 6 years; cases having durations of 6·6 to 7·0 inclusive would be considered as entering (or emerging) at an integral duration of 7 years; and cases entering (or emerging) precisely mid-way, or at 6·5 years, would be alternately classed as entrants (or as emergents) at integral durations of 6, and of 7 years. The operation, so far, deals solely with the amount of fractional exposure within each separate year of duration, and adopts convenient and average assumptions of equivalent integral durations, *but always so as strictly to preserve the incidence of the cases in their true years of duration.*

In carrying this method out in practice, the cards of the "survivors" at the commencement of the period of observation would be entered up with the nearest integral duration at entry, according to the above plan; the cards of the cases "existing" would similarly be entered up with the nearest integral duration at the close of the period; and the cards of the cases of withdrawal would be entered up with the nearest integral duration at withdrawal. As regards the deaths, a different course would be followed, in an investigation intended to deduce the rate of mortality; for, in order to give each case of death a full year's exposure in the year of death, the duration entered upon the cards must be the year of duration then current (that is, the curtate duration + 1) and not the nearest integral duration.

The cards are then sorted into original entrants, and "survivors"; the latter being then sorted, counted, and tabulated, according to the integral durations at entry, as recorded upon the cards; and the cards then combined, and re-sorted into cases existing, withdrawals, and deaths; and these again sorted, counted, and tabulated according to their recorded integral durations at exit.

The tabulation takes the form set out in Schedule (D).

I shall throughout employ the convenient symbols

$$(as),\ (ae),\ (aw),\ \text{and}\ (ad)$$

to indicate the cases surviving, existing, withdrawing, and dying, which are, by the Nearest Duration Method, referred to the *beginning* of the year of duration current at entry or at exit; and the symbols

$$(bs),\ (be),\ (bw),\ \text{and}\ (bd)$$

to indicate the cases referred to the *end* of the year of duration current at entry or at exit; the sums of these quantities being, of course, equal to

$$s,\ e,\ w,\ \text{and}\ d$$

in any given year of duration.

Thus, by the Nearest Duration Method, the number of "survivors" tabulated against duration t, which I shall call $\mathrm{s}_{[x]+t}$, is equal to

$$[(bs)_{[x]+t-1} + (as)_{[x]+t}] = \mathrm{s}_{[x]+t}$$

and similarly with the cases existing (e), and withdrawing (w); as shewn in the headings of columns (2), (3), and (4), of Schedule (D).

SCHEDULE (D).—OBSERVATION EXTENDING OVER

Table showing methods of deducing the Numbers Exposed to Risk, and the Fractional Exposures being taken to the nearest integer. (This method the Rate of Withdrawal).—Central Age at Entry (20).

Duration	Survivors	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	Mortality: Exposed	Mortality: Rate
t	$(bs)_{[x]+t-1} + (as)_{[x]+t} = s_{[x]+t}$	$(be)_{[x]+t-1} + (ae)_{[x]+t} = e_{[x]+t}$	$(bw)_{[x]+t-1} + (aw)_{[x]+t} = w_{[x]+t}$	$d_{[x]+t-1}$	$(e+w+d) = f_{[x]+t}$	$(s-f) = g_{[x]+t}$	$n_{[x]} + \Sigma_0^t(g) = E_{[x]+t}$	$\frac{d_{[x]+t}}{E_{[x]+t}} = q_{[x]+t}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
							$n_{[x]}$=421	
0	56	40	32	...	72	− 16	405	·00247
1	128	78	43	1	122	+ 6	411	·01217
2	114	47	42	5	94	+ 20	431	·00696
3	132	42	54	3	99	+ 33	464	·00216
4	123	70	51	1	122	+ 1	465	·00645
5	114	70	40	3	113	+ 1	466	·00644
6	95	67	40	3	110	− 15	451	...
7	72	68	24	...	92	− 20	431	·00926
8	93	82	29	4	115	− 22	409	·01222
9	80	85	24	5	114	− 34	375	·00267
10	69	77	18	1	96	− 27	348	·00862
11	64	63	21	3	87	− 23	325	·01846
12	30	50	12	6	68	− 38	287	·00697
13	16	73	12	2	87	− 71	216	·00926
14	17	61	12	2	75	− 58	158	·00633
15	14	49	7	1	57	− 43	115	·01739
16	11	50	1	2	53	− 42	73	...
17	15	19	4	...	23	− 8	65	...
18	16	14	4	...	18	− 2	63	...
19	3	11	2	...	13	− 10	53	...
20	6	11	2	...	13	− 7	46	...
21	10	8	2	...	10	0	46	·02174
22	15	14	1	1	16	− 1	45	...
23	17	11	1	...	12	+ 5	50	...
24	11	3	...	...	3	+ 8	58	...
25	10	6	...	...	6	+ 4	62	·01613
26	2	10	1	1	12	− 10	52	·01923
27	...	15	1	1	17	− 17	35	·02857
28	...	13	...	1	14	− 14	21	...
29	...	8	1	...	9	− 9	12	·16667
30	...	8	...	2	10	− 10	2	...
31	...	2	...	...	2	− 2	...	...
	1,333	1,225	481	48	1,754	−421	6,440	...

The facts being entered in columns (2) to (5), the sum of the numbers existing, withdrawing, and dying, gives the value of (**f**) in column (6); and deducting these values from the number of survivors in column (2), we deduce the values of (**g**) in column (7). Summing these values continuously, and including throughout the number of original entrants (n_x), we arrive in column (8) at the numbers exposed to the risk of death in each year of duration. Finally, dividing the value of $E_{[x]+t}$ (in column 8) into that of $d_{[x]+t}$ (on the line below) in column (5), we arrive at the value of $q_{[x]+t}$, the rate of mortality in the $(t+1)$th year of duration, in column (9). *See* Appendix (A), [Schedule (D)].

FIVE CALENDAR YEARS.—NEAREST DURATION METHOD.—SCHEDULE (D).

Rates (a) of Mortality, (b) of Withdrawal, in true years of duration; the involves a re-sorting throughout of the Withdrawals and Deaths, to deduce

With-drawals	Deaths	Total Decrement	Net Movement	Withdrawal		Dura-tion
				Exposed	Rate	
$w_{[x]+t-1}$	$(bd)_{[x]+t-1} + (ad)_{[x]+t} = d_{[x]+t}$	$(e+w+d) = f'_{[x]+t}$	$(s-f') = g'_{[x]+t}$	$n_{[x]} + \Sigma_0^t(g') = (wE)_{[x]+t}$	$\frac{w}{wE} = (wq)_{[x]+t}$	t
(10)	(11)	(12)	(13)	(14) $n_{[x]} = 421$	(15)	(16)
...	1	41	+ 15	436	·1170	0
51	1	130	− 2	434	·1083	1
47	6	100	+ 14	448	·0982	2
44	2	88	+ 44	492	·0915	3
45	3	118	+ 5	497	·1107	4
55	2	127	− 13	484	·0909	5
44	1	112	− 17	467	·0707	6
33	1	102	− 30	437	·0526	7
23	6	111	− 18	419	·0644	8
27	3	115	− 35	384	·0417	9
16	1	94	− 25	359	·0529	10
19	6	88	− 24	335	·0597	11
20	4	74	− 44	291	·0378	12
11	2	86	− 70	221	·0588	13
13	0	74	− 57	164	·0671	14
11	2	62	− 48	116	·0172	15
2	1	53	− 42	74	·0676	16
5	...	24	− 9	65	·0462	17
3	...	17	− 1	64	·0469	18
3	...	14	− 11	53	·0189	19
1	...	12	− 6	47	·0638	20
3	1	12	− 2	45	·0222	21
1	...	15	0	45	...	22
...	...	11	+ 6	51	·0196	23
1	...	4	+ 7	58	...	24
...	...	6	+ 4	62	·0161	25
1	2	13	− 11	51	·0196	26
1	...	16	− 16	35	...	27
...	1	14	− 14	21	·0476	28
1	2	11	− 11	10	...	29
...	...	8	− 8	2	...	30
...	...	2	− 2	...	...	31
481	48	1,754	−421	6,667	...	

It will be seen that I have, in the tabular arrangement of Schedule (D), attempted to improve and simplify the form as given by Dr. Sprague (*J.I.A.*, xxxi, 212), by setting the values of $E_{[x]+t}$ and of $q_{[x]+t}$ (instead of those of $E_{[x]+t-1}$ and $q_{[x]+t-1}$) opposite the duration t. The numbers in the column headed "deaths" must necessarily be the successive values of $d_{[x]+t-1}$; but there is no practical difficulty in dividing the value of $E_{[x]+t}$ in column (8) into that of $d_{[x]+t}$ on the line below in column (5); and the headings of the columns are throughout more symmetrical, as now submitted. The same

remarks will apply to the columns (10) to (15) (which I shall presently refer to) for deducing the numbers exposed to risk, and the rate of withdrawal.

The values of $q_{[x]+t}$ thus computed agree closely with those deduced by the Exact Duration and the Mean Duration Methods, in Schedules (B) and (C) respectively; and there is, so far, no "mixing-up" of the years of duration. It may at first sight appear that the methods of sorting and tabulation do, in fact, mix up the years of duration; for, as regards the cases surviving, existing, and withdrawing, the cases having durations (at entry or at exit) between $(t-\cdot 5)$ and t years, are throughout combined with those having durations between t and $(t+\cdot 5)$ years. Further consideration will, however, shew that it is quite immaterial, *for the purpose of deducing the numbers exposed to the risk of death,* whether we consider the survivors, combined as above, as entering at the end of the tth or at the beginning of the $(t+1)$th year of duration; and, similarly, it is quite immaterial whether we consider the cases existing and withdrawing as emerging at the end of the tth, or at the beginning of the $(t+1)$th year. The deaths are, in all cases, tabulated in their true years of duration; and the rates of mortality are thus deduced without any overlapping of those years.

When, however, it is desired to deduce the rate of *withdrawal* in years of duration, it is quite clear that the facts as tabulated in the first eight columns of Schedule (D) will not give the desired result. For the true estimation of the rate of withdrawal, it is necessary to give the cases of withdrawal a full year's exposure in the year of exit, and to give the death-cases only their true (or estimated) exposure up to the date of death. In column (4) of Schedule (D), however, the withdrawals are exposed only up to their (estimated) date of exit; while in column (5) the death-cases have a full year's exposure. Further, the number of withdrawals in column (4) includes all cases withdrawing between durations $(t-\cdot 5)$ and $(t+\cdot 5)$;* and it is clear that if these values be employed as a numerator in deducing the

* It would thus appear that the ratio of withdrawals as deduced from the values in column (4) would give some approximation to the *force of withdrawal;* and this suggestion, which appears to have been originally made by Mr. G. King, is referred to by Mr. Ryan (*J.I.A.*, xxxi, 310). I cannot, however, trace the original reference, attributed to Mr. King, in the pages of the *Journal.* The divisor $E_{[x]+t}$, in column (8), is not appropriate for deducing the true force of withdrawal (as is pointed out by Mr. Ryan, *loc. cit.*); and I have been unable, from the values given in columns (2) to (8) of Schedule (D), to deduce any expression which would give a true (or very approximate) representation of the rate of withdrawal in successive years of duration.

rates of withdrawal, the rate deduced will not truly represent that obtaining either in the tth or $(t+1)$th year of duration; and that the withdrawals of adjacent years of durations would be improperly combined in the expression for the rate of withdrawal so deduced.

It will, moreover, be seen that the facts as tabulated in columns (4) and (5) of the Schedule, give no means of introducing the necessary corrections in the amount of exposure, nor of scheduling the cases in their true years of duration.

Under this method, then, the only course available for accurately deducing the rates of withdrawal appears to be that of *re-sorting throughout the whole of the cases of withdrawal and death,* the former according to the year of duration current at exit (that is, the curtate duration $+1$), the latter according to the nearest integral duration at death. This is, of course, a laborious process. I have set out in columns (10) and (11) of Schedule (D), the results of this re-sorting.

Following, then, precisely the same course as in deducing the rate of mortality, we arrive at the modified values of $\mathbf{f}'$ $(=\mathrm{e}+w+\mathrm{d})$ in column (12), and of $\mathbf{g}'$ $(=\mathrm{s}-\mathbf{f}')$ in column (13); then summing the values of $\mathbf{g}'$ as before, and adding in the number of original entrants $(n_{[x]})$, we arrive in column (14), at the numbers exposed to the risk of withdrawal; and dividing $(w_{[x]+t})$ in column (10) by $(wE)_{[x]+t}$ in column (14), we deduce the true rate of withdrawal $(wq)_{[x]+t}$ in column (15).

These rates will be found to agree closely with those deduced in Schedules (B) and (C), by the Exact Duration and the Mean Duration Methods respectively.

By this second process of sorting and tabulation (which, as will be seen, nearly doubles the work involved) the rate of withdrawal can, then, be deduced in true years of duration.

Let us now investigate the constituent parts of the values tabulated in columns (4) and (10), and in columns (5) and (11) of Schedule (D), in which the withdrawals and deaths are set out in the forms required respectively for deducing the rate of mortality and of withdrawal.

We have, for the withdrawals in column (4)

$$[(bw)_{[x]+t-1}+(aw)_{[x]+t}]=\mathrm{w}_{[x]+t},$$

and for the withdrawals in column (10)

$$[(aw)_{[x]+t-1}+(bw)_{[x]+t-1}]=w_{[x]+t-1}.$$

Similarly, for the deaths in column (5), we have

$$[(ad)_{[x]+t-1}+(bd)_{[x]+t-1}]=d_{[x]+t-1},$$

and for the deaths in column (11)

$$[(bd)_{[x]+t-1}+(ad)_{[x]+t}]=\mathrm{d}_{[x]+t}.$$

These alternative expressions for the cases of withdrawal and of death can therefore be deduced by any process which shall set out separately the values of

$$(aw),\ (bw),\ (ad),\ (bd)$$

in successive years of duration. In other words, if the withdrawals and deaths are sorted *according to half-years of actual duration at exit,* the above expressions can be readily deduced.

In Schedule (E) I have thus sorted and tabulated the withdrawals and deaths, so that in column (4) are given the values of (aw), and (below them) of (bw), in each year of duration; while in column (5) are given the values of (ad) and of (bd), in each year of duration. The sorting into half-years at exit involves very little additional labour, for in arriving at the nearest integral duration, the half-year of exit necessarily comes under observation. Thus, the cards at nearest integral duration (t) are made up of two groups; (1) cases with durations from $(t-{\cdot}5)$ to t years inclusive; and (2) cases with durations from t to $(t+{\cdot}5)$ inclusive; and if the operator, instead of combining these groups, keeps them separate throughout, the cards are at once sorted in half-years of duration. Where the exact duration has not been already recorded upon the cards, they may be conveniently marked as follows:—

Duration over 7 years and less than $7\frac{1}{2}$ years (with one-half of the cases of exact duration $7\frac{1}{2}$ years).	7+
Duration over $7\frac{1}{2}$ years, up to and including 8 years (with one-half of the cases of exact duration $7\frac{1}{2}$ years).	8−

The values of (s) and (e) in columns (2) and (3) agree with those given in Schedule (D), and those of (f) in column (6) of Schedule (E) are arrived at by summing the values in columns (3), (4), and (5), thus:—

$$\mathrm{e}_{[x]+t}+\left\{\begin{matrix}(bw)_{[x]+t-1}\\(aw)_{[x]+t}\end{matrix}\right\}+\left\{\begin{matrix}(bd)_{[x]+t-1}\\(ad)_{[x]+t}\end{matrix}\right\}=(\mathrm{e}+\mathrm{w}+\mathrm{d})_{[x]+t}=\mathrm{f}_{[x]+t}$$

as indicated by the numbers coupled by brackets in the Schedule. In column (7) the values of g(=s−f) are set out, and in column (8) these values are summed vertically, including the number of original entrants $n_{[x]}$. The values thus arrived at are those of $\overline{E}_{[x]+t}$, the numbers exposed to risk, computed up to the date of death or withdrawal, the formula being (*see* Appendix A)

$$\overline{E}_{[x]+t}=n_{[x]}+\Sigma_0^t(g) \quad . \quad . \quad . \quad . \quad (11)$$

The numbers exposed to the risk of death and of withdrawal respectively are then deduced by the formulæ

$$E_{[x]+t}=\overline{E}_{[x]+t}+(ad)_{[x]+t}$$

$$(wE)_{[x]+t}=\overline{E}_{[x]+t}+(aw)_{[x]+t} \text{ (\textit{see} Appendix C).}$$

These numbers are tabulated in columns (9) and (12) respectively. In columns (10) and (13) I have set out (for the sake of clearness) the values of $d_{[x]+t}$ and of $w_{[x]+t}$, which represent simply the sums of the adjacent values tabulated in pairs in columns (5) and (4) respectively; and in columns (11) and (14) are deduced the values of $q_{[x]+t}$ and of $(wq)_{[x]+t}$. These rates, as well as the numbers exposed to risk in columns (9) and (12), are identical throughout with those given in Schedule (D).

It will be remarked that the experience dealt with in both schedules differs from that investigated by Dr. Sprague (*J.I.A.*, xxxi, 212) by including a body of survivors (s) in force at the commencement of the period of observation, instead of tracing the assurances from their original entry. The introduction of the survivors presents no special feature of difficulty. It must, however, be noted that the survivors who have, at entry on the period of observation, a duration of less than six months, will, by the rule of nearest duration, be classed as of "Duration 0" at entry, and will thus practically be included among the cases entering as new assurances during the period of observation. Thus, there were, in the experience here dealt with, 56 cases of survivors of less than six months' duration at the commencement of the period of observation. These 56 cases were included with the 421 cases of new entrants during the period, thus raising the total to 477.

On the other hand, there were, at the close of the period of observation, 40 cases existing with a duration of less than six months, and therefore classed as existing at "Duration 0": these 40 cases were altogether eliminated from the experience, thus

SCHEDULE (E).—OBSERVATION EXTENDING OVER

Table shewing methods of deducing the Numbers Exposed to Risk, and the Fractional Exposures being taken to the nearest integer. (Cases of Central Age at Entry (20).

Duration	Survivors	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	$n_{[x]} + \Sigma_0^t(g)$
t	$\left.\begin{matrix}(bs)_{[x]+t-1} \\ +(as)_{[x]+t}\end{matrix}\right\}$ $=s_{[x]+t}$	$\left.\begin{matrix}(be)_{[x]+t-1} \\ +(ae)_{[x]+t}\end{matrix}\right\}$ $=e_{[x]+t}$	$(aw)_{[x]+t}$ $(bw)_{[x]+t}$	$(ad)_{[x]+t}$ $(bd)_{[x]+t}$	$f_{[x]+t}$	$g_{[x]+t}$ $=(s-f)$	$\overline{E}_{[x]+t}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8) $n_{[x]}=421$
0	56	40	32	1	73	−17	404
1	128	78	19 24	0 1	122	+ 6	410
2	114	47	23 19	4 2	95	+19	429
3	132	42	25 29	1 1	98	+34	463
4	123	70	16 35	0 3	124	− 1	462
5	114	70	20 20	0 2	112	+ 2	464
6	95	67	24 16	1 ..	108	−13	451
7	72	68	17 7	.. 1	93	−21	430
8	93	82	16 13	3 3	117	−24	406
9	80	85	14 10	2 1	112	−32	374
10	69	77	6 12	0 1	96	−27	347
11	64	63	7 14	2 4	90	−26	321
12	30	50	6 6	2 2	66	−36	285
13	16	73	5 7	0 2	87	−71	214
14	17	61	6 6	0 0	73	−56	158
15	14	49	5 2	1 1	58	−44	114
16	11	50	0 1	1 ..	52	−41	73
17	15	19	4 0	.. ..	23	− 8	65
18	16	14	3 1	.. ..	18	− 2	63
19	3	11	2 0	.. ..	13	−10	53
20	6	11	1 1	.. ..	13	− 7	46
21	10	8	2 0	.. 1	11	− 1	45
22	15	14	1 ..	0 ..	15	0	45
23	17	11	.. 1	.. ..	12	+ 5	50
24	11	3	0 ..	.. ..	3	+ 8	58
25	10	6	.. 0	.. 0	6	+ 4	62
26	2	10	1 0	1 1	13	−11	51
27	..	15	1 ..	0 0	16	−16	35
28	..	13	.. 0	1 ..	14	−14	21
29	..	8	1 ..	.. 2	11	−11	10
30	..	8	.. ..	0 ..	8	− 8	2
31	..	2	.. ..	.. ..	2	− 2	..
	1,333	1,225	256 225	29 19	1,754	−421	6,411

Five Calendar Years.—Nearest Duration Method.—Schedule (E).

Rates, of Mortality and of Withdrawal, in true years of duration; the Withdrawal and Death sorted in half-years of duration at exit.—

Mortality			Withdrawals			Duration
Exposed	Deaths	Rate	Exposed	Withdrawals	Rate	
$E_{[x]+t} = \overline{E} + (ad)$	$d_{[x]+t}$	$q_{[x]+t} = \frac{d}{E}$	$(wE)_{[x]+t} = \overline{E} + (aw)$	$w_{[x]+t}$	$wq_{[x]+t} = \frac{w}{wE}$	t
(9)	(10)	(11)	(12)	(13)	(14)	(15)
405	1	·00247	436	51	·1170	0
411	5	·01217	434	47	·1083	1
431	3	·00696	448	44	·0982	2
464	1	·00216	492	45	·0915	3
465	3	·00645	497	55	·1107	4
466	3	·00644	484	44	·0909	5
451	..	..	467	33	·0707	6
431	4	·00926	437	23	·0526	7
409	5	·01222	419	27	·0644	8
375	1	·00267	384	16	·0417	9
348	3	·00862	359	19	·0529	10
325	6	·01846	335	20	·0597	11
287	2	·00697	291	11	·0378	12
216	2	·00926	221	13	·0588	13
158	1	·00633	164	11	·0671	14
115	2	·01739	116	2	·0172	15
73	..	..	74	5	·0676	16
65	..	..	65	3	·0462	17
63	..	..	64	3	·0469	18
53	..	..	53	1	·0189	19
46	..	..	47	3	·0638	20
46	1	·02174	45	1	·0222	21
45	..	..	45	..	..	22
50	..	..	51	1	·0196	23
58	..	..	58	..	..	24
62	1	·01613	62	1	·0161	25
52	1	·01923	51	1	·0196	26
35	1	·02857	35	..	..	27
21	..	..	21	1	·0476	28
12	2	·16667	10	..	..	29
2	..	..	2	..	..	30
..	..	..	..	..	..	31
6,440	48	..	6,667	481	..	

practically reducing the new entrants to 437. Of these, 32 withdrew, and one died, within six months of entry (that is at "Nearest Duration 0"); and the number exposed to risk in the first year of duration ($\bar{\mathrm{E}}_{[x]}$) was thus finally $437-(32+1)=404$.

I have, for the sake of clearness, stated these operations at length: but it is to be remarked that they are, throughout, deduced directly by the formulæ and methods adopted in Schedule (E), and *involve no exceptional treatment whatever in the first year of duration.* In this respect I venture to hope that they will be found to compare favourably with the alternative methods adopted by Dr. Sprague (*loc. cit.*); and I submit the method, illustrated in Schedule (E), as a ready and practical means of deducing the numbers exposed to risk, and the rates of mortality and of withdrawal in true years of duration, according to the Nearest Duration Method.

I have given in Appendix (A) an algebraical analysis of the Nearest Duration Method as thus applied in deducing the numbers exposed to risk, and the rates of death and of withdrawal. It is there shewn that, while, by the Exact Duration Method,

$$\bar{\mathrm{E}}_{[x]+t}=n_{[x]}+\Sigma_0^t(g)-g'_{[x]+t} \quad . \quad . \quad . \quad . \quad (4)$$

by the Nearest Duration Method,

$$\bar{\mathrm{E}}_{[x]+t}=n_{[x]}+\Sigma_0^t(\mathrm{g}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$=n_{[x]}+\Sigma_0^t(g)-(bg)_{[x]+t} \quad . \quad . \quad . \quad (12)$$

The expressions $n_{[x]}$ and $\Sigma_0^t(g)$ are identical by the two formulæ, and $\Sigma_0^t(g)$, in formula (4), is equal to

$$\Sigma_0^t\{s-(e+w+d)\}$$

and represents the number exposed to risk in the $(t+1)$th year, upon the assumption that the "survivors" are exposed to *a full year's risk* in that year, and that the cases "existing" and emerging are exposed to *no risk* in that year. The expression in the Exact Duration formula (4) above

$$-g'_{[x]+t}=(e'_{[x]+t}+w'_{[x]+t}+d'_{[x]+t}-s'_{[x]+t})$$

modifies and corrects the expression $\Sigma_0^t(g)$ by *adding the true fractional exposures* of the cases "existing" and emerging in the $(t+1)$th year, and *deducting the complemental fractional exposures* of the cases "surviving" in that year, and thus deduces the true value of

$$\bar{\mathrm{E}}_{[x]+t}=\Sigma_0^t(g)-g'_{[x]+t}.$$

Turning now to the Nearest Duration formula (12) above, we have the corrective expression $-(bg)_{[x]+t}$, which is equal to

$$(be)_{[x]+t}+(bw)_{[x]+t}+(bd)_{[x]+t}-(bs)_{[x]+t}$$

and this expression similarly modifies the value of $\Sigma_0^t(g)$, by *adding the integral durations assumed to be approximately equal to the true fractional exposures* of the cases "existing" and emerging in the $(t+1)$th year, and *deducting the integral durations assumed to be approximately equal to the complemental fractional exposures* of the cases "surviving" in that year.

The difference in the value of $E_{[x]+t}$ by the two methods is thus represented solely by the error introduced in computing the fractional exposures in the $(t+1)$th year at their nearest integral value; and there is no overlapping of the years of duration at any stage of the processes followed, as set out in Schedule (E).

Comparison of Aggregate Numbers Exposed to Risk.

Comparing now the numbers exposed to risk of death and of withdrawal, as deduced in Schedules (B), (C), and (D) or (E), it will be seen that the aggregate numbers are:

	$\Sigma(E)$	$\Sigma(wE)$
Exact Duration Method . . .	6,457·9	6,660·7
Mean Duration Method . . .	6,444·5	6,661·0
Nearest Duration Method . . .	6,440·	6,667·

The values in individual years of duration, both of the numbers exposed to risk and of the rates of mortality and withdrawal, are also practically identical throughout; and it is thus abundantly evident that, *in the investigation of this particular experience,* each of the three methods will give accurate and trustworthy results.

I now proceed to consider how far these conclusions will be modified, when consideration is given to the special characteristics of the experience here employed for purposes of illustration, and of that obtaining amongst assured lives generally.

Special characteristics of Illustrative Experience, as distinguished from those of assured lives generally.

This subject may be conveniently considered under two headings; (1) as to the actual or assumed age at entry (2) as to the distribution of entrants and emergents over the year of duration current at entry and exit.

(1) As to the Age at Entry.

It will have been seen that the experience here tabulated is set out throughout according to office ages at entry and years of duration. This is a very convenient, and seems to be the most appropriate, course, in a case such as that here dealt with, where the experience of an Association is investigated, and the results applied to the valuation of its own risks; for the rates, as deduced, can be applied to the computation of valuation factors tabulated in the same form; and the cases for valuation being similarly scheduled (according to office age at entry and years of duration as at date of valuation) these factors can be directly applied in valuing the risks, as is more fully shewn in Part II of the present paper. No assumptions are then involved as to the age at entry, nor does any question arise as to the ages at date of valuation. The only assumption made seems to be that the actual entry ages of the members in force at the date of valuation, bear the same relations, on the whole (whether in excess or defect) to their office ages, which obtained in the general body of lives investigated over the past quinquennium; and it does not appear to be material whether the office ages were, in point of fact, taken as at next, or last, or nearest birthdays, so long as the same general relations hold good throughout.

When, however, the case of lives in an assurance office is dealt with, and especially in the case where the aggregate experience of several distinct companies is under investigation, these considerations do not hold good; and the office age at entry appears to be no longer a safe or trustworthy basis. This would appear to be especially the case where the experience is deduced and tabulated in years of duration. Several suggestions have been made as to the most appropriate method of determining the ages at entry in such a case; and a brief consideration of some of the methods suggested may, perhaps, be useful.

There seems to be a general agreement as to the desirability, in the words of Mr. Whittall (*J.I.A.*, xxxi, 166) of adopting "that method of determining the ages at entry which most nearly agrees with the actual ages of the assured at entry." Among the methods that have been suggested are that of (1) "Nearest Age at Entry," called by Dr. Sprague, "commencing age," and adopted by him in an illustrative experience (given in *J.I.A.*, xxxi, 208); that of (2) "Mean Age at Entry," stated by Messrs. G. F. Hardy and Rothery (*J.I.A.*, xxvii, 165); and

that of (3) "Nearest Age at nearest 31 December," given by Mr. G. King (*J.I.A.*, xxvii, 218). These methods may be concisely stated, and their relative advantages briefly considered.

(1) The "*Nearest Entry Age*" is simply arrived at by taking the age as at the birthday nearest to the actual date at entry. The amount of divergence from the true age at entry cannot exceed six months, the greatest difference, of course, arising when the birthday precedes or follows the date of entry by an interval of half a year. This method gives results which compare very favourably with those deduced by other assumptions. It would, however, appear that where a large proportion of the lives have a tendency to enter near their next birthdays, the true ages at entry will on the whole be somewhat overstated by the Nearest Age Method, as all such cases will, by that method, be taken as at entry age next birthday. Judging, however, from the table given by Mr. Whittall (*J.I.A.*, xxxi, 187) of the experience of the Clerical, Medical, and General Office in this respect, as regards entrants in the two years 1853–4 and 1886–7, it may be considered that this tendency of the method in question to overstate the age at entry is not in practice sufficiently marked to affect the accuracy of the results.

(2) The "*Mean Age*" is arrived at by deducting the calendar year of birth from the calendar year of entry. Here the divergence from the true age at entry may be almost a year; the greatest difference arising, in excess, where the date of entry is at the beginning, and the date of birth at the end, of the calendar year (a very unusual case); and, in defect, where the birthday is at the beginning, and the date of entry at the end, of the calendar year (a not unusual case). Some objections to this method have been stated by Mr. Whittall and the late Mr. Sunderland (*J.I.A.*, xxvii, 191, 193); but judging from the table (already referred to) published by Mr. Whittall, it gives in practice very good results. It is, however, clearly inferior in point of accuracy to the method of nearest ages.

(3) The modification of the Nearest Age Method, as suggested by Mr. King, proceeds upon the plan of taking the "Age at Entry" as the age which falls *on the birthday nearest to the nearest* 31 *December at entry*. This takes practical effect by fixing the age at entry, in the case of entrants in the first half of a calendar year, at the birthday which falls in the year made up of the current and the preceding half-years; and, in the case of entrants in the second half of a calendar year, at the birthday

which falls in the year made up of the current and the following half-years.* The results may differ by almost a year from the true entry ages, the greatest difference arising, in excess, where the date of entry, in second half-year, immediately follows the birthday in the first half-year (a very unusual case); and, in defect, where the date of entry in first half-year immediately precedes the birthday in the second half year (a by no means unusual case). This method has, as I understand, been proposed particularly for the investigation of the experience of an office in an inter-valuation period, and not as specially suited for the aggregate experience of assured lives. It will probably, however, give on the whole good results: but is clearly inferior to the method of Nearest Ages.

In order to show graphically, and in a tabular form, the effect of adopting these three methods, I append a comparative statement of the results arrived at, on different assumptions as to the date of entry and the birthdays of the entrants. In column (1) I have graphically delineated the calendar year with quarterly or half-yearly divisions; and the letters (*e*) and (*b*) represent respectively the dates of entry and of the birthday, the incidence of each being supposed to occupy a mean position in the divisions in which they are respectively located. In column (2) I have set out the office age at entry next birthday, which I have taken as (x) throughout; and in column (3) is given the true average age at entry, as deduced from the mean positions of the date of entry and of the birthday in column (1). In columns (4), (6) and (8) I have given the "Age at Entry" as deduced by the "Mean Age" method, Mr. King's method, and the "Nearest Age" method; and in columns (5), (7) and (9) I have stated the divergence between the ages so estimated and the true ages as given in column (3). For convenience, the cases are separately tabulated and summarized where the date of entry respectively precedes and follows the birthday in the same calendar year.

* I cannot entirely agree with the conclusion of Mr. Whittall (*J.I.A.*, xxxi, 184), that this method may be classed as one of "mean ages"; and Mr. King has himself expressed his dissent from this view (*J.I.A.*, xxxi, 201). The method, as stated by Mr. King, gives, not true, but *modified* "mean ages", which coincide with those arrived at by deducting the year of birth from the year of entry, *in the case only where both years are reckoned from* 1 *July to* 30 *June.* If, however, the method be so applied as to deduce the ages which fall on the birthdays nearest to the nearest 30 *June* to the dates of entry (a modification contemplated by Mr. King, *J.I.A.*, xxvii, 218), the results will in that case precisely coincide with *true* "mean ages," computed by deducting the *calendar* years of birth from the *calendar* years of entry. I am also unable to agree with Mr. King that the error of age can never exceed six months. This seems to me to be demonstrably inaccurate, at least, as regards the age at entry, with which alone I am now concerned; and I can only suppose that Mr. King is referring to the average error, and not to individual cases of deviation.

Comparative Statement showing results of different methods of estimating Ages at Entry, according to the distribution of the Date of Entry, and the birthday, in the calendar year of entry.

Calendar Year: Showing assumed location of Dates of Entry and Birthdays in Quarters and Half-Years. (1)	Office Age next Birthday (2)	True Average Entry Age (3)	"Mean Age" Method: Entry Age (4)	"Mean Age" Method: Average Error (5)	Mr. King's Method: Entry Age (6)	Mr. King's Method: Average Error (7)	"Nearest Age" Method: Entry Age (8)	"Nearest Age" Method: Average Error (9)
	(1) Date of Entry preceding Birthday in Calendar Year.							
(1) e. b.	x	$x-\frac{1}{6}$	x	$+\frac{1}{6}$	x	$+\frac{1}{6}$	x	$+\frac{1}{6}$
(2) e. b.	x	$x-\frac{1}{6}$	x	$+\frac{1}{6}$	x	$+\frac{1}{6}$	x	$+\frac{1}{6}$
(3) e b	x	$x-\frac{1}{2}$	x	$+\frac{1}{2}$	$x-1$	$-\frac{1}{2}$	$\begin{Bmatrix}x\\x-1\end{Bmatrix}$	0
(4) e b	x	$x-\frac{3}{4}$	x	$+\frac{3}{4}$	$x-1$	$-\frac{1}{4}$	$x-1$	$-\frac{1}{4}$
(5) e b	x	$x-\frac{1}{4}$	x	$+\frac{1}{4}$	$x-1$	$-\frac{3}{4}$	x	$+\frac{1}{4}$
(6) e b	x	$x-\frac{1}{2}$	x	$+\frac{1}{2}$	$x-1$	$-\frac{1}{2}$	$\begin{Bmatrix}x\\x-1\end{Bmatrix}$	0
Total	$=6x$	$6x-2\frac{1}{3}$	$6x$	$+2\frac{1}{3}$	$6x-4$	$-1\frac{2}{3}$	$6x-2$	$+\frac{1}{3}$
	(2) Date of Entry following Birthday in Calendar Year.							
(7) b. e.	x	$x-\frac{5}{6}$	$x-1$	$-\frac{1}{6}$	$x-1$	$-\frac{1}{6}$	$x-1$	$-\frac{1}{6}$
(8) b. e.	x	$x-\frac{5}{6}$	$x-1$	$-\frac{1}{6}$	$x-1$	$-\frac{1}{6}$	$x-1$	$-\frac{1}{6}$
(9) b e	x	$x-\frac{1}{2}$	$x-1$	$-\frac{1}{2}$	x	$+\frac{1}{2}$	$\begin{Bmatrix}x-1\\x\end{Bmatrix}$	0
(10) b e	x	$x-\frac{1}{4}$	$x-1$	$-\frac{3}{4}$	x	$+\frac{1}{4}$	x	$+\frac{1}{4}$
(11) b e	x	$x-\frac{3}{4}$	$x-1$	$-\frac{1}{4}$	x	$+\frac{3}{4}$	$x-1$	$-\frac{1}{4}$
(12) b e	x	$x-\frac{1}{2}$	$x-1$	$-\frac{1}{2}$	x	$+\frac{1}{2}$	$\begin{Bmatrix}x-1\\x\end{Bmatrix}$	0
Total	$=6x$	$6x-3\frac{2}{3}$	$6x-6$	$-2\frac{1}{3}$	$6x-2$	$+1\frac{2}{3}$	$6x-4$	$-\frac{1}{3}$
Grand Total	$=12x$	$12x-6$	$12x-6$	0	$12x-6$	0	$12x-6$	0

The aggregate results over the 12 cases cited, are identical by the three methods, giving an average entry age of $(x-\frac{1}{2})$, which agrees also with the true average age at entry of the 12 cases. But, looking at the cases in detail, and bearing in mind that the 12 typical examples given will not be equally numerous in actual experience, the great superiority of the "Nearest Age" method will be readily seen by comparison of the column (9) with columns (5) and (7). It may, however, be added that the cases which

will be likely to be relatively most numerous (at least, in the ordinary case where premiums are quoted for integral ages only) will be those numbered (1), (2), (5), and (10); and it will be noted that in these four typical cases, the "Nearest Age" method always overstates the age at entry (by taking it at next birthday); while both the Mean Method and that of Mr. King overstate the age in three of the typical examples, and understate it in the fourth case.

I cannot here consider this question further; but these brief remarks and the appended statement may, perhaps, be of some service in elucidating the points involved.

(2) Distribution of Entrants and Emergents over years of duration current at entry and at exit.

Reverting now to the comparative statement of aggregate numbers exposed to risk, as deduced by the three methods exemplified in Schedules (B), (C), and (D) or (E), it is to be remarked that, in the particular experience here investigated, the members' contributions are payable at short periodical intervals; and there is no apparent reason why cases should not enter upon observation, emerge during the period, or survive to its close, so as to be equally distributed over the years of duration. In point of fact, it will be seen, upon reference to Schedule (A) (page 9), that the fractional exposures, whether in individual years of duration, or in the aggregate, do not materially differ from one-half of the corresponding number of cases. The ratios of the aggregate fractional exposures to the aggregate number of cases are as follow:—

"Survivors" (at entry)	$= \cdot 566$
"Existing" (at close of period)	$= \cdot 570$
Withdrawals	$= \cdot 529$
Deaths	$= \cdot 506$

If it be borne in mind that all the cases enter, and all (excepting the deaths) emerge, on the first day of a calendar month, and that (assuming an equal distribution of entrants, and of emergents, throughout the year) the average exposure during the year would thus be equal to

$$\frac{12+11+10+9+8+7+6+5+4+3+2+1}{12}$$

$= 6 \cdot 5$ months $= \cdot 5417$ of a year; it will be seen that the cases, whether of entry or of emergence, are practically distributed equally over the year of duration.

The particular experience here investigated, differs, in this

respect, materially from that usually obtaining among assured lives, and it may be of interest to give some consideration to this point. In the case of a period of observation limited by calendar years, the survivors at the commencement will (among assured lives generally) tend to come under observation, and the cases existing will tend to terminate their experience, in the first half of their then current years of duration; while the withdrawals will tend to congregate towards the end of the year of duration current at exit.

The average fractional duration of the withdrawals will depend to some extent upon the proportions of cases effected at yearly, half-yearly, and quarterly premiums. If, however, the reasonable assumption be made that cases effected at half-yearly premiums are equally likely to withdraw (by lapse) at the date of the first or second half-yearly payment, and that cases effected at quarterly premiums are equally likely to withdraw (by lapse) at the date of the first, second, third or fourth quarterly payment, the average fractional duration of the withdrawals will be, for half-yearly cases

$$\tfrac{1}{2}(\tfrac{1}{2}+1)=\tfrac{3}{4}$$

and for quarterly cases,

$$\tfrac{1}{4}(\tfrac{1}{4}+\tfrac{1}{2}+\tfrac{3}{4}+1)=\tfrac{5}{8}$$

so that there will be a marked tendency (upon the above assumptions) apart from the actual proportions of yearly, half-yearly and quarterly cases, for the withdrawals by lapse to fall, on the average, in the latter half of the year of duration.

Mr. Chatham, in his recent Messenger Prize Essay, assumes (*J.I.A.*, xxxii, 412) that the proportions of the yearly, half-yearly, and quarterly cases will be respectively 75 per-cent, 20 per-cent, and 5 per-cent. After some consideration and enquiry, I have preferred, for present purposes, to assume that the proportions are as $62\frac{1}{2}$ per-cent, $32\frac{1}{2}$ per-cent, and 5 per-cent; that is, that of every 400 cases, 250 are effected at yearly payments, 130 at half-yearly payments, and 20 at quarterly payments, of premium. Assuming that 400 withdrawals take place, in a given year of duration, in the above proportions, and that the withdrawals of half-yearly and quarterly cases are equally likely to fall at either of the two half-yearly, or at any of the four quarterly, epochs for payment, the aggregate exposure of the 400 cases in their year of withdrawal would be equal to

$$250+(\cdot 75\times 130)+(\cdot 625\times 20)=360,$$

thus giving an average exposure of $\frac{9}{10}$ths of a year, for each

case. If the cases were in the proportions assumed by Mr. Chatham, the aggregate exposure of the 400 withdrawals would be, upon the same assumptions, 372·5 years, with an average exposure of ·93125 of a year for each case.

In order to ascertain the effect of employing the methods of Mean Duration, and of Nearest Duration, in the case of an experience where the average exposure of the entrants and the emergents in the years of duration current at entry or at exit, materially varied from half a year, I have applied the Nearest Duration Method to the same illustrative experience which has been already here employed; but upon the assumption that the surviving entrants, at the commencement of the period of observation, and the cases existing at its close, had, on the whole, completed *one-fourth* only of the year of duration current at the commencement, and at the end, of the period of observation, respectively; also that the withdrawals during the period had, on the whole, an average duration of *nine-tenths* of a year in the year of duration current at exit.

These proportions are not intended to represent the actual, or most probable, experience of any company or group of companies; but are adopted merely for purposes of illustration. These assumptions do not, of course, in any case affect the total number of cases surviving, existing or withdrawing, as observed in successive years of duration, but solely affect their distribution over the years of duration current at entry and at exit.

The results are set out in tabular form in Schedule (F); and it will be seen that the above assumptions are given effect to, upon the Nearest Duration Method, by so dividing the total cases in any year of duration that $(as)=3(bs)$; $(ae)=3(be)$; and $(bw)=9(aw)$; these values being computed to the nearest integer. The death cases were tabulated throughout in half-years, as actually experienced.

To illustrate more clearly the methods followed, and the effects of the assumptions made as to the distribution of the cases, I have tabulated the cases surviving, existing, withdrawing and dying, in half-years of duration throughout. It will be understood, however, that as regards the "surviving" and "existing," in columns (2) and (3), it is not necessary to sort or tabulate the cases in half-years, for the purpose of computing the numbers exposed to risk, or the rates of mortality and withdrawal.

The number exposed to risk in each year of duration ($\bar{E}_{[x]+t}$) is computed by formula (11), and the values of $E_{[x]+t}$ and of

$(wE)_{[x]+t}$, and of $q_{[x]+t}$ and $(wq)_{[x]+t}$, are deduced as in Schedule (E), pp. 26, 27.

If now the numbers exposed to risk of death and of withdrawals, as arrived at in Schedule (F), are compared with those deduced in Schedule (E) upon the basis of a (practically) uniform distribution of the entrants and emergents throughout the year, it will be seen that the aggregate results are as follows :—

NEAREST DURATION METHOD.

	Σ(E)	$\Sigma(wE)$
Uniform Distribution (Schedule E) .	6,440	6,667
Assumed Distribution (Schedule F) .	6,664	6,683
Difference	+224	+16

Thus the numbers exposed to the risk of death are, in this particular experience, considerably increased when the special assumptions as to distribution are given effect to; while the numbers exposed to the risk of withdrawal, are (in this particular case) not substantially varied.

It will be readily seen that the difference between the numbers exposed to the risk of death, as deduced (by the Nearest Duration Method) from the assumed distribution of entrants and withdrawals, and as deduced (by the same method) from their uniform distribution, may be expressed as equal to

$$\left\{\tfrac{1}{4}(e-s)-[(be)-(bs)]\right\}+\left\{\frac{9w}{10}-(bw)\right\}$$

which expression holds good in any given year of duration, or in the aggregate. Taking aggregate values, we have,

$$\left\{\tfrac{1}{4}(1225-1333)-(668-711)^*\right\}+\left\{\frac{9}{10}\times 481-225\right\}$$
$$=(16+207\cdot 9)=223\cdot 9,$$

which agrees closely with the actual difference of the aggregate numbers.

Similarly, the difference in the numbers exposed to the risk of withdrawal is equal to

$$\tfrac{1}{4}(e-s)-[(be)-(bs)]$$
$$=\tfrac{1}{4}(1225-1333)-(668-711)^*=16.$$

Here the distribution of the death-cases, in the year of death, does not affect the numbers exposed to the risk of death; and similarly the distribution of the withdrawals, in the year of

* The values of (be) and (bs) are not separately stated in Schedule (E); but I have taken them from working sheets, which include the values throughout in half-years of duration.

SCHEDULE (F).—OBSERVATION EXTENDING OVER
ASSUMED DISTRIBUTION OF

Table showing methods of deducing the Numbers Exposed to Risk, and the Fractional Exposures being taken to the

Duration	Survivors	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	$n_{[x]} + \Sigma_0^t(g)$
t	$\cdot 25 s_{[x]+t-1}$ $\cdot 75 s_{[x]+t}$ $= s_{[x]+t}$	$\cdot 25 e_{[x]+t-1}$ $\cdot 75 e_{[x]+t}$ $= e_{[x]+t}$	$\cdot 9 w_{[x]+t-1}$ $\cdot 1 w_{[x]+t}$ $= w_{[x]+t}$	$(bd)_{[x]+t-1}$ $(ad)_{[x]+t}$ $= d_{[x]+t}$	$(e + w + d) = f_{[x]+t}$	$(s - f) = g_{[x]+t}$	$= \bar{E}_{[x]+t}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
							$n_{[x]} = 421$
0	103	61	5	1	67	+ 36	457
	34	21	46	0			
1	76	62	5	1	135	− 25	432
	25	21	42	4			
2	99	5	5	2	79	+ 45	477
	33	2	39	1			
3	90	54	5	1	102	+ 21	498
	30	18	40	0			
4	97	56	6	3	123	+ 4	502
	32	19	49	0			
5	76	52	4	2	126	− 18	484
	25	17	40	1			
6	58	46	3	..	107	− 24	460
	19	16	30	..			
7	67	55	2	1	104	− 18	442
	22	19	21	3			
8	65	61	3	3	110	− 23	419
	22	21	24	2			
9	57	67	2	1	117	− 38	381
	19	22	14	0			
10	48	52	2	1	91	− 24	357
	16	17	17	2			
11	36	36	2	4	78	− 26	331
	12	12	18	2			
12	11	49	1	2	84	− 61	270
	4	16	10	0			
13	14	52	1	2	81	− 63	207
	5	17	12	0			
14	14	44	1	0	74	− 55	152
	5	15	10	1			
15	7	35	0	1	62	− 50	102
	3	11	2	1			
16	10	26	1	..	41	− 28	74
	3	9	4	..			
17	11	8	0	..	21	− 7	67
	4	3	3	..			
18	7	10	0	..	16	− 5	62
	3	3	3	..			
19	3	12	0	..	18	− 12	50
	1	4	1	..			
20	7	5	0	..	10	− 2	48
	2	2	3	..			
21	8	8	0	1	14	− 4	44
	2	3	1	0			
22	11	8	..	..	12	+ 1	45
	4	2	..	..			
23	12	7	0	..	9	+ 7	52
	4	2	1	..			
24	8	3	..	..	6	+ 6	58
	2	1	..	..			
25	5	7	0	0	8	− 1	57
	2	2	1	1			
26	..	8	0	1	13	− 11	46
	..	2	1	0			
27	..	9	..	0	12	− 12	34
	..	3	..	1			
28	..	11	0	..	15	− 15	19
	..	3	1	..			
29	..	6	..	2	12	− 12	7
	..	2	..	0			
30	..	4	..	.	6	− 6	1
	..	1	..	..			
31	..	..	..	..	1	− 1	..
	..	..	..	..			
	1,000	919	48	29	..	..	..
	333	306	433	19	1,754	−421	6,635

FIVE CALENDAR YEARS.—NEAREST DURATION METHOD.—SCHEDULE (F). ENTRANTS AND WITHDRAWALS.

Rates, of Mortality and of Withdrawal, in true years of duration; the nearest integer.—Central Age at Entry (20).

MORTALITY			WITHDRAWAL			Duration
Exposed	Deaths	Rate	Exposed	Withdrawals	Rate	
$E_{[x]+t} = \bar{E} + (ad)$	$d_{[x]+t}$	$q_{[x]+t} = \frac{d}{\bar{E}}$	$(wE)_{[x]+t} = \bar{E} + (aw)$	$w_{[x]+t}$	$(wq)_{[x]+t} = \frac{w}{(wE)}$	t
(9)	(10)	(11)	(12)	(13)	(14)	(15)
458	1	·00218	462	51	·1104	0
433	5	·01155	437	47	·1075	1
479	3	·00626	482	44	·0913	2
499	1	·00200	503	45	·0895	3
505	3	·00594	508	55	·1083	4
486	3	·00617	488	44	·0902	5
460	..	..	463	33	·0713	6
443	4	·00903	444	23	·0518	7
422	5	·01185	422	27	·0640	8
382	1	·00262	383	16	·0418	9
358	3	·00837	359	19	·0529	10
335	6	·01719	333	20	·0600	11
272	2	·00735	271	11	·0406	12
209	2	·00957	208	13	·0625	13
152	1	·00658	153	11	·0719	14
103	2	·01942	102	2	·0196	15
74	..	..	75	5	·0667	16
67	..	..	67	3	·0448	17
62	..	..	62	3	·0484	18
50	..	..	50	1	·0200	19
48	..	..	48	3	·0625	20
45	1	·02222	44	1	·0227	21
45	..	..	45	..	..	22
52	..	..	52	1	·0192	23
58	..	..	58	..	..	24
57	1	·01754	57	1	·0175	25
47	1	·02128	46	1	·0217	26
34	1	·02941	34	..	..	27
19	..	..	19	1	·0526	28
9	2	·22222	7	..	..	29
1	..	..	1	..	..	30
..	..	..	..	..	..	31
6,664	48	..	6,683	481	..	

withdrawal, does not affect the numbers exposed to the risk of withdrawal. This would of course be anticipated, and evidently arises from the fact that the deaths and withdrawals respectively are given a full year's exposure in their year of exit.

It will also be remarked that the difference in the aggregate numbers exposed to the risk of death and of withdrawal, is considerably affected by the fact that the numbers existing (e) and surviving (s) are approximately equal, and nearly balance one another. If the experience were such that the value of $(s-e)$ was relatively large (as, for example, where the cases are under observation from original entry up to a fixed date, and the "survivors" wholly disappear), the results above shown would be materially varied.

A comparison of the values of q and of (wq), in columns (10) and (12) of Schedule (F), with those given in columns (11) and (14) of Schedule (E) will show the effect, upon the rates of mortality and withdrawal, of the assumed variation in the distributions of the entrants and emergents. It will be seen that the rate of mortality is, under the assumed distribution, lower in the first 12 years of duration, and throughout higher, from the 13th year of duration to the end of the table; while the rates of withdrawal, although somewhat lower in the first six years of duration, do not, upon the whole, show any very marked deviations.

Let us now consider the effect of the distribution of entrants and emergents, in the case where the Mean Duration Method is applied. It is evident that this method, being based upon the assumption of an equal distribution of the entrants and emergents, will give precisely the same results, whatever the actual or assumed distribution may be; and that those results will, in the case of this particular experience, be those shown in Schedule (C), pp. 16, 17. Comparing, then, the numbers exposed to risk, as given in Schedules (C) and (F), we find that the aggregate values are as follow:—

Assumed Distribution of Entrants and Emergents.

	Σ(E)	$\Sigma(wE)$
Mean Duration Method (Schedule C) .	6,444·5	6,661·0
Nearest Duration Method (Schedule F)	6,664·	6,683·
Difference	+219·5	+22·0

Here, the difference in the numbers exposed to the risk of death is evidently equal to

$$\tfrac{1}{4}(s-e)+\left(\frac{9w}{10}-\frac{w}{2}\right)$$
$$=27+192\cdot4=219\cdot4$$

and the difference in the numbers exposed to the risk of withdrawal is equal to

$$\tfrac{1}{4}(s-e)+\left[(bd)-\frac{d}{2}\right]$$
$$=27+[19-24]=22$$

The remarks made above as to the counter-balancing effects upon these results of the nearly equal values of (e) and (s) will also apply here.

It is thus evident that the Mean Duration Method gives, as might have been anticipated, erroneous results as regards the numbers exposed to risk, and the resulting rates of mortality and of withdrawal, in the case supposed where the average fractional exposures of the entrants and emergents materially differs from half a year.

Special Suitability of Nearest Duration Method for the Experience of Assured Lives generally.

The Nearest Duration Method, as here applied (in Schedule F) to the case of an assumed distribution of entrants and emergents, cannot be compared with the Exact Duration Method, as in this illustrative case the exact durations have not been ascertained, or rather have been assumed to be faithfully reproduced by the Nearest Duration Method. There can, however, be no doubt of the special suitability of this method to the experience of assured lives; for this method gives effect, to a very great extent, to the actual average durations of the assurances in their several years of duration, and that by a sort of automatic reference to the beginning or end of those years. As compared with the Mean Duration Method, which gives an average of half-a-year's exposure to all entrants and emergents (and thus deduces a result which can hardly fail to be erroneous in the case of assured lives) the Nearest Duration Method has manifest advantages.

This method also seems to me to be greatly superior to the method [suggested and illustrated by Mr. Ryan (*J.I.A.*, xxvi, 259-264)], of taking a *constant ratio* of the entrants and emergents, to represent their assumed fractional exposure in all years of duration. The Nearest Duration Method, by giving approximate

effect to the true fractional exposures in each year of duration, will allow for any marked deviation from the general average ratio in individual years; and thus possesses an elasticity which compares very favourably with the rigidity of the "Constant Ratio" method.

Let us now consider briefly the probable deviations of the Nearest Duration Method from the true exposures, as regards the several cases of withdrawals (1) by lapse, (2) by surrender, (3) by miscellaneous causes; also of cases "surviving" and "existing"; and cases of death.

Taking first the cases of withdrawal (1) by lapse, it will be borne in mind that cases of non-renewal of premium, occurring at the end of the year of duration (whether in respect of yearly, half-yearly, or quarterly premiums), will, in all cases, be given their true integral exposures; also, that cases of non-renewal occurring in the middle of the year of duration (whether in respect of half-yearly or quarterly premiums) will, by the system of alternate reference to the beginning and end of the year of exit, be also given their true exposures, taken one with another. There only remain the quarterly cases of non-renewal at the first and third quarters of the year of duration; and those occurring at the first quarter (and referred by the Nearest Duration Method to the beginning of the year) may fairly be considered to be balanced by those occurring in the third quarter (and referred to the end of the year).

As regards (2) surrenders, there will be a certain error in the estimated exposures in respect of the period intervening between the date of surrender and the next following renewal date. This period is, in practice, usually found to be very short; and any error will be much reduced by the fact that the intervening period is understated, if the surrender take place in the first half of the year of exit, and overstated if it occur in the second half. If there be (a) yearly cases, (b) half-yearly cases, and (c) quarterly cases, surrendered in any year of duration; if $\left(\frac{1}{m}\right)$ represent the period intervening between surrender and the next renewal date; and if it be assumed that a surrender is equally likely to take place before any of the four quarterly renewal dates, or before either of the half-yearly renewal dates; it can readily be shown that the aggregate exposures of the yearly cases surrendering are overstated by $\frac{a}{m}$ years; and that the aggregate exposures of the

half-yearly and quarterly cases are respectively understated by $b\left(\frac{1}{4}-\frac{1}{m}\right)$years and by $c\left(\frac{1}{8}-\frac{1}{m}\right)$ years. The aggregate error would thus amount to

$$\frac{a}{m}-b\left(\frac{1}{4}-\frac{1}{m}\right)-c\left(\frac{1}{8}-\frac{1}{m}\right)$$

and the average error would be found by dividing this expression by $(a+b+c)$. If we assume (as before) that the cases are in the proportions of 250 yearly, 130 half-yearly, and 20 quarterly, and that the average interval between surrender and the next renewal date is one month, the above expression becomes

$$\frac{250}{12}-\frac{130}{6}-\frac{20}{24}=-\frac{10}{6}$$

showing an *aggregate* error (for 400 cases) of $1\frac{2}{3}$ years' exposures; and an *average* error of $\frac{1}{240}$th of a year, or say $1\frac{1}{2}$ days, in each case. The remarkable power of the Nearest Duration Method, in adapting itself to the conditions of actual practice, could, perhaps, hardly be better exemplified.

The miscellaneous cases of withdrawal (3) include maturity of endowment assurances; expiration of term assurances; cancelment by forfeiture, &c. These cases would not be relatively numerous; and while the majority would usually fall at the end of the year current at exit, the remainder may fairly be considered as likely to be equally distributed over that year.

As regards cases of surviving entrants, they will, as already stated, tend to enter upon observation in the first half of the year of duration current at entry; at least, in the usual case, where the commencement of the period of observation is also the commencement of a financial year of the office. These cases will, by the Nearest Duration Method, be given an exposure for the whole of the year of duration current at entry, which will be in excess of the true exposures.

Upon the other hand, the cases existing at the close of the period of observation will, on the whole, tend to pass out of observation in the first half of the year of duration then current; and such cases will, by the Nearest Duration Method, be given no exposure in that year of duration, thus involving an understatement of the true exposures. There is thus a compensating influence at work, in reduction of the balance of error; and where the number of cases of surviving entrants does not greatly differ from that of the cases "existing", the ultimate error may be

expected to be relatively small. In the case, however, of an experience where there are no "survivors", there would be no such compensating influence; and the cases "existing" would, as a whole, have their exposures understated by the Nearest Duration Method.

Finally, as regards the cases of death. Upon the usual assumption of an equal distribution of the deaths over the year of duration current at death, the average exposure would be half a year; and this would be correctly tabulated, taking one case with another, both by the Mean Duration Method, and the Nearest Duration Method. (See, however, Appendix D).

(B) Period of Observation Limited by Years of Duration. Illustrative Experience.

We have hitherto investigated the case where the period of observation extends over an integral number of calendar years, so that survivors enter at fractional durations at the commencement, and cases exist, also at fractional durations, at the close. If now the experience is so investigated that the period of observation runs concurrently with years of duration, that is, so that the "survivors" come under observation from the commencement, and the cases "existing" are traced up to the close, of a year of duration, the formulæ and the tabular operations are throughout very much simplified. As these are precisely the conditions under which the New Institute Experience is being investigated, the cases being traced from their policy anniversary in the calendar year 1863 (or from subsequent entry) to their policy anniversary in 1893 (or previous exit), I have deemed it useful to investigate this condition of things somewhat in detail.

For this purpose I eliminated, from the illustrative experience previously employed (1) all cases emerging (by death or withdrawal) prior to their policy anniversaries in 1888; (2) all cases entering in the year 1892; while cases emerging (by death or withdrawal) during the year 1892, and subsequently to their policy anniversaries in that year, were treated as "existing." There were thus eliminated 65 "survivors" and 89 original entrants, leaving 1,268 and 332 respectively; of which 1,184 existed at their policy anniversaries in 1892, 385 withdrew during the period of observation, and 31 died.

The period of observation thus extended over four years of duration, from the policy anniversary in 1888 (in the case of "survivors") to the policy anniversary in 1892 (in the case of

"existing"); and included all cases of entry, withdrawal, and death falling between those anniversaries.

I have, in Appendix (B), investigated the modifications introduced, in the formulæ already given, to meet the particular case of an experience limited by years of duration, in the several cases where the Exact Duration Method, the Mean Duration Method, and the Nearest Duration Method, are adopted.

Exact Duration Method.

I have computed, in the appended Schedule (G), the numbers exposed to risk, and deduced the rates of mortality and withdrawal, by the formulæ

$$E_{[x]+t} = \Sigma_0^t(G) + (w' + d)_{[x]+t} \quad . \quad . \quad . \quad (18)$$

and

$$(wE)_{[x]+t} = \Sigma_0^t(G) + (w + d')_{[x]+t} \quad . \quad . \quad . \quad (21)$$

where

$$G_{[x]+t} = [s^1 - (e^1 + w + d)]_{[x]+t}$$

and s^1 and e^1 indicate the numbers respectively surviving and existing at the integral duration of t years; the withdrawals (w) and the deaths (d) being tabulated as before, according to their curtate durations at exit.

Upon comparing Schedule (G) with Schedule (B)* it will be seen that the tabular operations are much simplified, in the case here investigated, where the period of observation is limited by years of duration.—In Schedule (G), fractional expressions enter only into column (4) of withdrawals, and column (5) of deaths; and by the formulæ given above, the values of E and of (wE) are deduced by a direct process, and one not involving the calculation of the function $\overline{E}$. It will also be noted that the "new entrants" (332 in number) during the period of observation here conveniently figure as "surviving entrants" at precise age (x); and that the table, as a whole, is thus rendered more symmetrical and convenient.

This Schedule (G) appears to me to present, in a very simple form, the whole of the tabular work involved in deducing the numbers exposed to risk of death and of withdrawal, and the resulting rates, in true years of duration, and with exact fractional exposures. It need hardly be added that the values of q and of (wq) deduced in columns (11) and (14) cannot properly be compared with those set out in columns (15) and (17) of Schedule (B), as the period of observation, and the data involved, are not identical in the two cases.

* See pp. 12, 13.

Schedule (G).—Observation Extending over

Table showing methods of deducing the Numbers Exposed to Risk, and the with exact Fractional Exposures.

Curtate Duration t	Surviving $s^1{}_{[x]+t}$	Existing $e^1{}_{[x]+t}$	Withdrawals $w_{[x]+t}$ $w'_{[x]+t}$	Deaths $d_{[x]+t}$ $d'_{[x]+t}$	Total Decrement (e^1+w+d) $=F_{[x]+t}$	Net Movement (s^1-F) $=G_{[x]+t}$	$\Sigma_0^t(G)$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	332	..	35 16·6	.. ..	35	+297	297
1	128	85	40 21·9	3 2·5	128	0	297
2	93	7	36 19·1	2 0·4	45	+ 48	345
3	124	75	40 18·5	1 0·5	116	+ 8	353
4	116	80	42 18·2	2 0·8	124	− 8	345
5	121	77	32 18·5	2 0·5	111	+ 10	355
6	96	62	29 15·9	.. ..	91	+ 5	360
7	74	78	17 10·3	2 0·8	97	− 23	337
8	84	84	24 14·5	2 1·3	110	− 26	311
9	82	91	11 5·5	1 0·3	103	− 21	290
10	74	74	15 7·9	2 1·1	91	− 17	273
11	63	50	17 8·9	6 3·3	73	− 10	263
12	48	65	10 5·6	1 0·5	76	− 28	235
13	13	70	12 5·6	1 0·1	83	− 70	165
14	18	61	9 5·1	1 0·6	71	− 53	112
15	19	47	1 0·3	1 0·1	49	− 30	82
16	9	36	4 2·6	.. .	40	− 31	51
17	13	11	3 2·3	.. ..	14	− 1	50
18	14	14	.. ..	.. ..	14	0	50
19	9	16	1 0·8	.. ..	17	− 8	42
20	4	7	3 1·8	.. ..	10	− 6	36
21	9	12	1 0·6	.. ..	13	− 4	32
22	10	10	.. ..	.. ..	10	0	32
23	15	9	.. ..	.. ..	9	+ 6	38
24	15	4	.. ..	.. ..	4	+ 11	49
25	10	9	1 0·1	1 0·6	11	− 1	48
26	7	10	1 0·8	1 0·5	12	− 5	43
27	..	12	.. ..	1 0·9	13	− 13	30
28	..	14	1 0·9	.. ..	15	− 15	15
29	..	9	.. ..	1 0·2	10	− 10	5
30	..	5	..	..	5	− 5	..
	1,600 ..	1,184 ..	385 202·3	31 15·0	1,600 ..	0 ..	4,941 ..

Four Years of Duration.—Exact Duration Method.—Schedule (G).
Rates, of Mortality and of Withdrawal, in true years of duration; and —Central Age at Entry (20).

$(w'_{[x]+t} + d_{[x]+t})$	Mortality: Exposed $E_{[x]+t} = \Sigma_0^t(G) + (w'+d)_{[x]+t}$	Mortality: Rate $q_{[x]+t} = \frac{d}{E}$	$(w_{[x]+t} + d'_{[x]+t})$	Withdrawal: Exposed $(wE)_{[x]+t} = \Sigma_0^t(G) + (w+d')_{[x]+t}$	Withdrawal: Rate $wq_{[x]+t} = \frac{w}{(wE)}$	Curtate Duration t
(9)	(10)	(11)	(12)	(13)	(14)	(15)
16·6	313·6	..	35·0	332·0	·1054	0
24·9	321·9	·00932	42·5	339·5	·1178	1
21·1	366·1	·00546	36·4	381·4	·0944	2
19·5	372·5	·00269	40·5	393·5	·1017	3
20·2	365·2	·00548	42·8	387·8	·1083	4
20·5	375·5	·00533	32·5	387·5	·0826	5
15·9	375·9	..	29·0	389·0	·0746	6
12·3	349·3	·00573	17 8	354·8	·0480	7
16·5	327·5	·00611	25·3	336·3	·0714	8
6·5	296·5	·00337	11·3	301·3	·0365	9
9·9	282·9	·00707	16·1	289·1	·0519	10
14·9	277·9	·02159	20·3	283·3	·0600	11
6·6	241·6	·00414	10·5	245·5	·0407	12
6·6	171·6	·00583	12·1	177·1	·0677	13
6·1	118·1	·00847	9·6	121·6	·0740	14
1·3	83·3	·01200	1·1	83·1	·0120	15
2 6	53·6	..	4·0	55·0	·0727	16
2·3	52·3	..	3·0	53·0	·0566	17
..	50·0	..	..	50·0	..	18
0·8	42·8	..	1·0	43·0	·0233	19
1·8	37·8	..	3·0	39·0	·0769	20
0·6	32·6	..	1·0	33·0	·0303	21
..	32·0	..	..	32·0	..	22
..	38·0	..	..	38·0	..	23
..	49·0	..	..	49·0	..	24
1·1	49·1	·02037	1·6	49·6	·0202	25
1·8	44·8	·02232	1·5	44·5	·0225	26
1·0	31·0	·03226	0·9	30·9	..	27
0·9	15·9	..	1·0	16·0	·0625	28
1·0	6·0	·16667	0·2	5·2	..	29
..	..	..	..	..	..	30
233·3	5,174·3	..	400·0	5341·0	..	

Mean Duration Method.

In this case, the formulæ for the numbers exposed to risk become respectively

$$E_{[x]+t} = \Sigma_0^t(G) + \left(\frac{w}{2} + d\right)_{[x]+t} \quad . \quad . \quad . \quad (24)$$

and

$$(wE)_{[x]+t} = \Sigma_0^t(G) + \left(w + \frac{d}{2}\right)_{[x]+t} \quad . \quad . \quad . \quad (27)$$

and the tabular results are set out in Schedule (H).

Schedule (H).—Observation Extending over

Table showing methods of deducing the Numbers Exposed to Risk, and the mean or average Fractional Exposures.

Curtate Duration	Surviving	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	$\Sigma_0^t(G)$
t	$s^1_{[x]+t}$	$e^1_{[x]+t}$	$w_{[x]+t}$	$d_{[x]+t}$	(e^1+w+d) $=F_{[x]+t}$	(s^1-F) $=G_{[x]+t}$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	332	...	35	...	35	+297	297
1	128	85	40	3	128	0	297
2	93	7	36	2	45	+ 48	345
3	124	75	40	1	116	+ 8	353
4	116	80	42	2	124	− 8	345
5	121	77	32	2	111	+ 10	355
6	96	62	29	...	91	+ 5	360
7	74	78	17	2	97	− 23	337
8	84	84	24	2	110	− 26	311
9	82	91	11	1	103	− 21	290
10	74	74	15	2	91	− 17	273
11	63	50	17	6	73	− 10	263
12	48	65	10	1	76	− 28	235
13	13	70	12	1	83	− 70	165
14	18	61	9	1	71	− 53	112
15	19	47	1	1	49	− 30	82
16	9	36	4	...	40	− 31	51
17	13	11	3	...	14	− 1	50
18	14	14	...	...	14	0	50
19	9	16	1	...	17	− 8	42
20	4	7	3	...	10	− 6	36
21	9	12	1	...	13	− 4	32
22	10	10	...	...	10	0	32
23	15	9	...	...	9	+ 6	38
24	15	4	...	...	4	+ 11	49
25	10	9	1	1	11	− 1	48
26	7	10	1	1	12	− 5	43
27	...	12	...	1	13	− 13	30
28	...	14	1	...	15	− 15	15
29	...	9	...	1	10	− 10	5
30	...	5	...	...	5	− 5	...
	1,600	1,184	385	31	1,600	0	4,941

Here again the operations are simplified as compared with those shown in Schedule (C),* and the values of E and of (wE) are deduced by direct processes. This method is especially rapid in its operations, and is very suitable for the investigation of the experience of a body of lives, where the average fractional exposure at exit does not greatly differ from half a year.

* See pp. 16, 17.

Four Years of Duration.—Mean Duration Method.—Schedule (H).

Rates of Mortality and of Withdrawal, in true years of duration; and with —Central Age at Entry (20).

$\left(\frac{w_{[x]+t}}{2} + d_{[x]+t}\right)$	Mortality: Exposed $E_{[x]+t} = \Sigma_0^t(G) + \left(\frac{w}{2}+d\right)$	Mortality: Rate $q_{[x]+t} = \frac{d}{E}$	$\left(w_{[x]+t} + \frac{d_{[x]+t}}{2}\right)$	Withdrawal: Exposed $(wE)_{[x]+t} = \Sigma_0^t(G) + \left(w+\frac{d}{2}\right)$	Withdrawal: Rate $(wq)_{[x]+t} = \frac{w}{(wE)}$	Curtate Duration t
(9)	(10)	(11)	(12)	(13)	(14)	(15)
17·5	314·5	...	35·0	332·0	·1054	0
23·0	320·0	·00938	41·5	338·5	·1182	1
20·0	365·0	·00548	37·0	382·0	·0942	2
21·0	374·0	·00267	40·5	393·5	·1017	3
23·0	368·0	·00544	43·0	388·0	·1082	4
18·0	373·0	·00536	33·0	388·0	·0825	5
14·5	374·5	...	29·0	389 0	·0746	6
10·5	347·5	·00576	18·0	355·0	·0479	7
14·0	325·0	·00615	25·0	336·0	·0714	8
6·5	296·5	·00337	11·5	301·5	·0365	9
9·5	282 5	·00708	16·0	289·0	·0519	10
14·5	277·5	·02163	20·0	283·0	·0601	11
6·0	241·0	·00415	10·5	245·5	·0407	12
7·0	172·0	·00581	12·5	177·5	·0676	13
5·5	117·5	·00851	9·5	121·5	·0741	14
1·5	83·5	·01198	1·5	83·5	·0120	15
2·0	53·0	...	4·0	55·0	·0727	16
1·5	51·5	...	3·0	53·0	·0566	17
...	50·0	...	...	50·0	...	18
0·5	42·5	...	1·0	43·0	·0233	19
1·5	37·5	...	3·0	39·0	·0769	20
0·5	32·5	...	1·0	33·0	·0303	21
...	32·0	...	...	32·0	...	22
...	38·0	...	...	38·0	...	23
...	49·0	...	...	49·0	...	24
1·5	49·5	·02020	1·5	49·5	·0202	25
1·5	44·5	·02247	1·5	44·5	·0225	26
1·0	31·0	·03226	0·5	30·5	...	27
0·5	15·5	...	1·0	16·0	·0625	28
1·0	6·0	·16667	0·5	5·5	...	29
...	...	...	...	...	...	30
223·5	5,164·5	...	400 5	5,341·5	...	

Nearest Duration Method.

In this case the formulæ become respectively

$$E_{[x]+t}=\Sigma_0^t(G)+(ad)_{[x]+t} \quad . \quad . \quad . \quad (29)$$

and

$$(wE)_{[x]+t}=\Sigma_0^t(G)+(aw)_{[x]+t} \quad . \quad . \quad . \quad (30)$$

where

$$G_{[x]+t}=[s^1-(e^1+\mathrm{w}+\mathrm{d})]_{[x]+t}$$

w and d representing, as before, the numbers of withdrawals and deaths as tabulated by the Nearest Duration Method.

The values are tabulated in Schedule (J).

The cases "surviving" and "existing" are respectively set out, in columns (2) and (3), at their true integral durations, the new entrants $n_{[x]}$ figuring as $s_{[x]}$, and in columns (4) and (5) the withdrawals and deaths are respectively tabulated, according to their half-years of duration at exit. The numbers existing, withdrawing, and dying, as set out in columns (3), (4) and (5), are then summed (as indicated by the brackets) in column (6); their sum being deducted from the numbers "surviving" in column (2), and the difference entered in column (7). The values thus arrived at, continuously summed in column (8), give the values of Σ_0^t $(G)=\bar{E}_{[x]+t}$ the numbers exposed to risk up to the dates of death or withdrawal; from which the values of E and of (wE) are deduced, by the above formulæ, in columns (9) and (12) respectively. In columns (10) and (13) the total number of deaths and of withdrawals are tabulated for convenience, these numbers being simply the sums of those set out, in adjacent half-years, in columns (5) and (4) respectively. Finally, in columns (11) and (14), the rates of mortality and of withdrawal are computed, for each year of duration.

I have explained these operations in detail, because this method appears to me to be, upon the whole, that best suited for practically dealing with a large body of assured lives, in such a way as to deduce, by a very simple and rapid series of operations, the rates of mortality and of withdrawal, in their true years of duration.

Comparing Schedule (J) with Schedule (E),* where the observation was limited by calendar years, it will be seen that the tabular operations are practically identical throughout, and as

* See pp. 26, 27.

$\Sigma_0^t\,(G)=\overline{E}_{[x]+t}$ in this case, this latter function is necessarily computed (in both Schedules) by the ordinary processes followed.

Comparison of Aggregate Numbers Exposed to Risk.

The aggregate numbers exposed to risk of death and of withdrawal, as deduced by the three methods, are as follow:

	$\Sigma(E)$	$\Sigma(wE)$
Exact Duration Method (Schedule G) .	5,174·3	5,341·0
Mean Duration Method (Schedule H) .	5,164·5	5,341·5
Nearest Duration Method (Schedule J)	5,148·	5,337·

These results are practically identical, the greatest difference not exceeding 1 in 200, or one-half per-cent.

The rates of mortality q and of withdrawal (wq) also give closely identical results, in each year of duration, as deduced by the three methods.

Assumed Distribution of Emergents over Years of Duration current at Exit.

In Schedule (K) I have tabulated, by the Nearest Duration Method, the cases included in the period of observation as limited by years of duration, upon the basis of an assumed distribution of the withdrawals over the years of duration current at exit; and, as before, I have assumed that the withdrawals, taken one with another, are exposed for nine-tenths of the year of exit. The methods followed, and the tabular arrangement, are identical with those set out in Schedule (J), the only difference in the data being in column (4), where the withdrawals are set out in half-years of duration, according to their assumed distribution, and so that

$$(bw)=9(aw),$$

throughout.

If we now compare Schedule (J) with Schedule (K), we shall see the results, upon the numbers exposed to risk, and upon the rates of death and withdrawal, of a variation in the distribution of the withdrawals over their year of exit. It will be seen that the aggregate numbers exposed to risk are as follow:—

SCHEDULE (J).—OBSERVATION EXTENDING OVER

Table showing methods of deducing the Numbers Exposed to Risk, and the Fractional Exposures being taken to the

Duration	Surviving	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	
t	$s^1_{[x]+t}$	$e^1_{[x]+t}$	$\left.\begin{matrix}(bw)_{[x]+t-1}\\(aw)_{[x]+t}\end{matrix}\right\} = \mathrm{w}_{[x]+t}$	$\left.\begin{matrix}(bd)_{[x]+t-1}\\(ad)_{[x]+t}\end{matrix}\right\} = \mathrm{d}_{[x]+t}$	$(e^1+\mathrm{w}+\mathrm{d}) = \mathrm{F}_{[x]+t}$	$(s^1-\mathrm{F}) = \mathrm{G}_{[x]+t}$	$\overline{\mathrm{E}}_{[x]+t} = \Sigma_0^t(\mathrm{G})$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	332	..	21	..	21	+311	311
1	128	85	14 / 23	.. / 0	122	+ 6	317
2	93	7	17 / 19	3 / 2	48	+ 45	362
3	124	75	17 / 24	0 / 1	117	+ 7	369
4	116	80	16 / 28	0 / 2	126	− 10	359
5	121	77	14 / 14	0 / 2	107	+ 14	373
6	96	62	18 / 16	0 / ..	96	0	373
7	74	78	13 / 5	.. / 1	97	− 23	350
8	84	84	12 / 11	1 / 1	109	− 25	325
9	82	91	13 / 7	1 / 1	113	− 31	294
10	74	74	4 / 8	0 / 1	87	− 13	281
11	63	50	7 / 11	1 / 4	73	− 10	271
12	48	65	6 / 6	2 / 1	80	− 32	239
13	13	70	4 / 7	0 / 1	82	− 69	170
14	18	61	5 / 5	0 / 0	71	− 53	117
15	19	47	4 / 1	1 / 1	54	− 35	82
16	9	36	0 / 1	0 / ..	37	− 28	54
17	13	11	3 / 0	.. / ..	14	− 1	53
18	14	14	3 / ..	.. / ..	17	− 3	50
19	9	16	.. / 0	.. / ..	16	− 7	43
20	4	7	1 / 1	.. / ..	9	− 5	38
21	9	12	2 / 0	.. / ..	14	− 5	33
22	10	10	1 / ..	.. / ..	11	− 1	32
23	15	9	.. / ..	.. / ..	9	+ 6	38
24	15	4	.. / ..	.. / ..	4	+ 11	49
25	10	9	.. / 1	.. / 0	10	0	49
26	7	10	0 / 0	1 / 1	12	− 5	44
27	..	12	1 / ..	0 / 0	13	− 13	31
28	..	14	.. / 0	1 / ..	15	− 15	16
29	..	9	1 / ..	.. / 1	11	− 11	5
30	..	5	.. / ..	0 / ..	5	− 5	..
	1,600	1,184	209	20	1,600	0	5,128
	..	..	176	11	..	..	..

FOUR YEARS OF DURATION.—NEAREST DURATION METHOD.—SCHEDULE (J).

Rates of Mortality and of Withdrawal, in true years of duration; the nearest integer.—Central Age at Entry (20).

Mortality			Withdrawal			Duration
Exposed	Deaths	Rate	Exposed	Withdrawals	Rate	Duration
$E_{[x]+t} = \Sigma_0^t(G) + (ad)$	$d_{[x]+t}$	$q_{[x]+t} = \frac{d}{E}$	$(wE)_{[x]+t} = \Sigma_0^t(G) + (aw)$	$w_{[x]+t}$	$(wq)_{[x]+t} = \frac{w}{(wE)}$	t
(9)	(10)	(11)	(12)	(13)	(14)	(15)
311	..	..	332	35	·1054	0
317	3	·00946	340	40	·1177	1
364	2	·00549	381	36	·0945	2
370	1	·00270	393	40	·1018	3
361	2	·00554	387	42	·1085	4
375	2	·00533	387	32	·0827	5
373	..	..	389	29	·0746	6
351	2	·00570	355	17	·0479	7
326	2	·00614	336	24	·0714	8
295	1	·00339	301	11	·0366	9
282	2	·00709	289	15	·0519	10
275	6	·02182	282	17	·0603	11
240	1	·00417	245	10	·0408	12
171	1	·00585	177	12	·0678	13
117	1	·00855	122	9	·0738	14
83	1	·01205	83	1	·0120	15
54	..	..	55	4	·0727	16
53	..	..	53	3	·0566	17
50	..	..	50	..	..	18
43	..	..	43	1	·0233	19
38	..	..	39	3	·0770	20
33	..	..	33	1	·0303	21
32	..	..	32	..	..	22
38	..	..	38	..	..	23
49	..	..	49	..	..	24
49	1	·02041	50	1	·0200	25
45	1	·02222	44	1	·0227	26
31	1	·03226	31	..	..	27
16	..	..	16	1	·0625	28
6	1	·16667	5	..	..	29
..	..	..	..	..	..	30
5,148	31	..	5,337	385	..	
..	..	..	..	..	..	

Schedule (K).—Observation Extending over
Assumed Distribution

Table showing methods of deducing the Numbers Exposed to Risk, and the Fractional Exposures being taken to the nearest

Duration	Survivors	Existing	Withdrawals	Deaths	Total Decrement	Net Movement	
t	$s^1_{[x]+t}$	$e^1_{[x]+t}$	$\left.\begin{matrix}\cdot 9 w_{[x]+t-1}\\ \cdot 1 w_{[x]+t}\end{matrix}\right\} = \mathrm{w}_{[x]+t}$	$\left.\begin{matrix}(bd)_{[x]+t-1}\\ (ad)_{[x]+t}\end{matrix}\right\} = \mathrm{d}_{[x]+t}$	$e^1 + \mathrm{w} + \mathrm{d} = \mathrm{F}_{[x]+t}$	$(s^1 - \mathrm{F}) = \mathrm{G}_{[x]+t}$	$\overline{\mathrm{E}}_{[x]+t} = \Sigma({}_0{}^t\mathrm{G})$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	332	..	4 31	.. ..	4	+ 328	328
1	128	85	4 36	0 3	120	+ 8	336
2	93	7	4 32	2 0	52	+ 41	377
3	124	75	4 36	1 0	112	+ 12	389
4	116	80	4 38	2 0	122	− 6	383
5	121	77	4 28	2 0	121	0	383
6	96	62	3 26	.. ..	93	+ 3	386
7	74	78	2 15	1 1	107	− 33	353
8	84	84	2 22	1 1	103	− 19	334
9	82	91	1 10	1 0	116	− 34	300
10	74	74	1 14	1 1	86	− 12	288
11	63	50	2 15	4 2	71	− 8	280
12	48	65	1 9	1 0	84	− 36	244
13	13	70	1 11	1 0	81	− 68	176
14	18	61	1 8	0 1	73	− 55	121
15	19	47	0 1	1 0	57	− 38	83
16	9	36	0 4	.. ..	37	− 28	55
17	13	11	0 3	.. ..	15	− 2	53
18	14	14	.. ..	.. ..	17	− 3	50
19	9	16	0 1	.. ..	16	− 7	43
20	4	7	0 3	.. ..	8	− 4	39
21	9	12	0 1	.. ..	15	− 6	33
22	10	10	.. ..	.. ..	11	− 1	32
23	15	9	.. ..	.. ..	9	+ 6	38
24	15	4	.. ..	.. ..	4	+ 11	49
25	10	9	0 1	0 1	9	+ 1	50
26	7	10	0 1	1 0	13	− 6	44
27	..	12	.. ..	0 1	13	− 13	31
28	..	14	0 1	.. ..	15	− 15	16
29	..	9	.. ..	1 0	11	− 11	5
30	..	5	.. ..	.. ..	5	− 5	..
	1,600 ..	1,184 ..	38 347	20 11	1,600 ..	.. 0	.. 5,299

FOUR YEARS OF DURATION.—NEAREST DURATION METHOD.—SCHEDULE (K). OF WITHDRAWALS.

Rates of Mortality and of Withdrawal, in true years of duration; the integer.—Central Age at Entry (20).

MORTALITY			WITHDRAWAL			Duration
Exposed	Deaths	Rate	Exposed	Withdrawals	Rate	
$E_{[x]+t} = \bar{E} + (ad)$	$d_{[x]+t}$	$q_{[x]+t} = \frac{d}{E}$	$(wE)_{[x]+t} = \bar{E} + (aw)$	$w_{[x]+t}$	$(wq)_{[x]+t} = \frac{w}{(wE)}$	t
(9)	(10)	(11)	(12)	(13)	(14)	(15)
328	..	..	332	35	·1054	0
336	3	·00893	340	40	·1177	1
379	2	·00528	381	36	·0945	2
390	1	·00256	393	40	·1018	3
385	2	·00519	387	42	·1085	4
385	2	·00519	387	32	·0827	5
386	..	..	389	29	·0746	6
354	2	·00565	355	17	·0479	7
335	2	·00597	336	24	·0714	8
301	1	·00332	301	11	·0366	9
289	2	·00692	289	15	·0519	10
284	6	·02113	282	17	·0603	11
245	1	·00408	245	10	·0408	12
177	1	·00565	177	12	·0678	13
121	1	·00826	122	9	·0738	14
84	1	·01190	83	1	·0120	15
55	..	..	55	4	·0727	16
53	..	..	53	3	·0566	17
50	..	..	50	..	..	18
43	..	..	43	1	·0233	19
39	..	..	39	3	·0770	20
33	..	..	33	1	·0303	21
32	..	..	32	..	..	22
38	..	..	38	..	..	23
49	..	..	49	..	..	24
50	1	·02000	50	1	·0200	25
45	1	·02222	44	1	·0227	26
31	1	·03226	31	..	..	27
16	..	..	16	1	·0625	28
6	1	·16667	5	..	..	29
..	..	..	..	..	..	30
5,319	31	..	5,337	385	..	

Nearest Duration Method.

	Σ(E)	Σ(wE)
Uniform Distribution (Schedule J) . .	5,148	5,337
Assumed Distribution (Schedule K) .	5,319	5,337
Difference	+171	0

Here the difference in the numbers exposed to risk of death is evidently equal to

$$\frac{9w}{10}-(bw)=346{\cdot}5-176=170{\cdot}5.$$

and it will be seen that the rate of mortality q as set out in column (11) of Schedule (K), is throughout lower than (or equal to) that tabulated in the corresponding column of Schedule (J).

As regards the rate of withdrawal, the numbers exposed to risk, and the resulting rates, columns (12) and (14), are throughout identical in Schedules (J) and (K), and are thus unaffected by the distribution of the withdrawals over the year of duration current at exit. This is, of course, self-evident, the withdrawals being credited with a full year's exposure in their year of exit; but it has, I think, been sometimes overlooked in discussions as to the (assumed) effect upon the rate of withdrawal of the distribution of the withdrawals over the year of duration current at exit.

Comparing now the results of Schedule (K) with those of Schedule (H), we shall see the results of adopting the Mean Duration Method, where the fractional exposure of the withdrawals in their year of exit differs materially from half a year. The following are the aggregate numbers exposed to risk.

Assumed Distribution of Withdrawals.

	Σ(E)	Σ(wE)
Mean Duration Method (Schedule H) .	5,164·5	5,341·5
Nearest Duration Method (Schedule K)	5·319	5,337
Difference	+154·5	−4·5

Here the difference in the numbers exposed to the risk of death is evidently equal to

$$\frac{9w}{10}-\frac{w}{2}=346{\cdot}5-192{\cdot}5=154{\cdot}0$$

and the rates of mortality by the Nearest Duration Method (Schedule K) are throughout lower than (or equal to) those deduced by the Mean Duration Method.

As regards the number exposed to the risk of withdrawal, the difference is limited to the estimated fractional exposures of the death cases by the two methods, and is equal to

$$(bd)-\frac{d}{2}=11-15{\cdot}5=-4{\cdot}5.$$

The numbers exposed to risk, and the rates, of withdrawal, are thus practically identical throughout.

It is thus again shown that the actual distribution of the withdrawals over their year of exit has a somewhat material effect upon the rates of mortality experienced, and in cases where the average fractional exposure of the withdrawals differs materially from half a year, the Nearest Duration Method may be expected to give a more faithful representation of the rates of death, than that which would be indicated by the Mean Duration Method.

Comparative Summary of Operations under the Exact Duration, the Mean Duration, and the Nearest Duration, Methods. Scheme of Notation.

In the appended Comparative Statement I have tried to set out a concise summary of the successive operations, under each of the three methods here discussed, for deducing the numbers exposed to risk, and the rates, of mortality and withdrawal; also the number of columns involved; the extent to which + and − signs, and fractions, enter into the computations; and the assumptions made as to the incidence of entrants and emergents in the years of duration current at entry and exit.

I also append a complete "Scheme of Notation", in which I have, for convenience of reference, brought together in a general view the several distinctive symbols used throughout Part I of this essay, according to the limitation of the period of observation and the different methods employed, as illustrated in the tabular Schedules (B) to (K).

In selecting the several symbols for the functions or quantities involved, I have tried (1) to depart as little as possible from those adopted and sanctioned by previous writers; (2) to employ symbols which should (as far as practicable) graphically suggest the functions or quantities which they represent; and (3) to distinguish by a variation of type those quantities, which, while functionally similar, differ in detail or in value when applied or computed according to a particular method.

COMPARATIVE

Of the successive Operations required for the computation of the Numbers of Duration:—by the application of the Exact Duration Method,

Under all three methods, the cards are first sorted according to original age at entry; (1) the separation into (*a*) new entrants, (*b*) surviving entrants; (2) the counting and operations in respect of each age (or group of ages) at entry:—

Method	"Surviving Entrants" and Cases "Existing"		Withdrawals and Deaths	
	Duration Recorded on Cards	Cards Sorted and Tabulated	Duration Recorded on Cards	Cards Sorted and Tabulated
(1)	(2)	(3)	(4)	(5)
				(A) Period of Observation Limited
Exact Duration	Exact fractional duration at entry, or at exit	According to curtate* duration at entry, or at exit (1) Number of cases (2) Aggregate fractional exposures	Exact fractional duration at exit	According to curtate* duration at exit (1) Number of cases (2) Aggregate fractional exposures
Mean Duration	Curtate* duration at entry, or at exit	According to curtate* duration at entry, or at exit (1) Number of cases	Curtate* duration at entry, or at exit	According to curtate* duration at exit (1) Number of cases
Nearest Duration	Nearest integral duration, at entry or at exit†	According to nearest integral duration† (1) Number of cases	Half-year of duration at exit†	According to half-year of duration of exit† (1) Number of cases in each half-year
				(B) Period of Observation Limited
Exact Duration	Integral duration at entry, or at exit	According to integral duration recorded on cards (1) Number of cases	Exact fractional duration at exit	According to curtate* duration (1) Number of cases (2) Aggregate fractional exposures
Mean Duration	Integral duration at entry, or at exit	According to integral duration recorded on cards (1) Number of cases	Curtate* duration at exit	According to curtate* duration (1) Number of cases
Nearest Duration	Integral duration at entry, or at exit	According to integral duration recorded on cards (1) Number of cases	Half-year of duration at exit†	According to half-year of duration at exit† (1) Number of cases in each half-year

* Integral durations of $(t+1)$ years being treated

† Durations of $(t+\frac{1}{2})$ being treated as of durations

‡ Columns in which figures are duplicated being

STATEMENT

Exposed to Risk, and the Rates of Mortality and Withdrawal, in Years the Mean Duration Method, and the Nearest Duration Method.

and the preliminary operations at each age at entry (or group of entry ages), are recording of the total number of new entrants. The following are the successive

No. of Columns‡ involved in tabulation of No. Exposed to Risk — Of Death (6)	No. of Columns‡ involved in tabulation of No. Exposed to Risk — Of Death and Withdrawal (7)	No. of Cols. involving attention to + and − signs (8)	Fractions introduced (9)	Assumptions made as to incidence of Entrants or Emergents in years of Duration current at Entrance or Exit (10)	Type of Operation in Schedule (11)	Method (12)
BY CALENDAR YEARS.						
15	16	4	Yes	None.	(B)	EXACT DURATION
10	11	4	·5 only	That the average fractional exposure of entrants and emergents = ·5	(C)	MEAN DURATION
10	11	2	None	(1) As regards cases entering and emerging exactly in middle or at end of year:—None (2) As regards cases entering and emerging at other points:—that the errors involved by reference to the nearest end of year counterbalance one another	(E), (F)	NEAREST DURATION
BY YEARS OF DURATION.						
11	13	2	Yes	None.	(G)	EXACT DURATION
9	11	2	·5 only	That the average exposure of the emergents at exit = ·5	(H)	MEAN DURATION
10	11	2	None	(1) As regards cases emerging exactly in middle or at end of year:—None (2) As regards cases emerging at other points:—that the errors involved by reference to the nearest end of year counterbalance one another.	(J), (K)	NEAREST DURATION

as of "curtate" duration (t), and "fractional exposure", (1·0).
(t), and ($t+1$), alternately.
each counted as two.

Scheme of Notation

Class of Observation, and Methods Employed	Schedule	Survivors	Existing
		Cases as tabulated,	
(A) Period of Observation limited by Calendar Years.			
(1) Exact Duration Method—	(B)		
Duration (t) as tabulated	...	curtate*	curtate*
Cases as tabulated	...	$s_{[x]+t}$	$e_{[x]+t}$
Aggregate Fractional Exposures	...	$s'_{[x]+t}$	$e'_{[x]+t}$
(2) Mean Duration Method—	(C)		
Duration (t) as tabulated	...	curtate*	curtate*
Cases as tabulated	...	$s_{[x]+t}$	$e_{[x]+t}$
(3) Nearest Duration Method—	(D) (a)		
(a) *For Mortality only*—			
Duration (t) as tabulated	...	nearest integral	nearest integral
Cases as tabulated	..	$\mathrm{s}_{[x]+t}=(bs)_{[x]+t-1}+(as)_{[x]+t}$	$\mathrm{e}_{[x]+t}=(be)_{[x]+t-1}+(ae)_{[x]+t}$
(b) *For Withdrawal only*—	(D) (b)		
Duration (t) as tabulated	...	nearest integral	nearest integral
Cases as tabulated	...	$\mathrm{s}_{[x]+t}=(bs)_{[x]+t-1}+(as)_{[x]+t}$	$\mathrm{e}_{[x]+t}=(be)_{[x]+t-1}+(ae)_{[x]+t}$
(c) *For Mortality and Withdrawal*—	(E),(F)		
Reference to beginning or end of Year of Duration	...	$(as)_{[x]+t}$ $(bs)_{[x]+t}$	$(ae)_{[x]+t}$ $(be)_{[x]+t}$
Duration (t) as tabulated	...	nearest integral	nearest integral
Cases as tabulated	...	$\mathrm{s}_{[x]+t}=(bs)_{[x]+t-1}+(as)_{[x]+t}$	$\mathrm{e}_{[x]+t}=(be)_{[x]+t-1}+(ae)_{[x]+t}$
(B) Period of Observation Limited by Years of Duration.			
(1) Exact Duration Method—	(G)		
Duration (t) as tabulated	...	true integral	true integral
Cases as tabulated	...	$s^1_{[x]+t}$	$e^1_{[x]+t}$
Aggregate Fractional Exposures	..	...	...
(2) Mean Duration Method—	(H)		
Duration (t) as tabulated	...	true integral	true integral
Cases as tabulated	...	$s^1_{[x]+t}$	$e^1_{[x]+t}$
(3) Nearest Duration Method—	(J),(K)		
Reference to beginning or end of Year of Duration	...	... ..	... ...
Duration (t) as tabulated	...	true integral	true integral
Cases as tabulated	..	$s^1_{[x]+t}$	$e^1_{[x]+t}$

* Integral durations of $(t+1)$ years being considered as of

Symbols *common to*

Numbers exposed to Risk—	
Of Death	$\mathrm{E}_{[x]+t}$
Of Withdrawal	$(wE)_{[x]+t}$
Of Death or Withdrawal	$\overline{\mathrm{E}}_{[x]+t}$
New Entrants at age $[x]$	$n_{[x]}$

employed in Part I.

Withdrawals	Deaths	Total Decrement	Net Movement
at Age at Entry $[x]$, and Duration (t).			
curtate* $w_{[x]+t}$ $w'_{[x]+t}$	curtate* $d_{[x]+t}$ $d'_{[x]+t}$	... $f_{[x]+t}$ $=(e+w+d)_{[x]+t}$ $f'_{[x]+t}$ $=(e'+w'+d')_{[x+t]}$	... $g_{[x]+t}$ $=(s-f)_{[x]+t}$ $g'_{[x]+t}$ $=(s'-f')_{[x]+t}$
curtate* $w_{[x]+t}$	curtate* $d_{[x]+t}$	... $f_{[x]+t}$ $=(e+w+d)_{[x]+t}$	... $g_{[x]+t}$ $=(s-f)_{[x]+t}$
nearest integral $\mathrm{w}_{[x]+t}=$ $(bw)_{[x]+t-1}+(aw)_{[x]+}$	current year $d_{[x]+t-1}$	... $\mathrm{f}_{[x]+t}$ $=(\mathrm{e}+\mathrm{w}+d)$	... $\mathrm{g}_{[x]+t}$ $=(\mathrm{s}-\mathrm{f})_{[x]+t}$
current year $w_{[x]+t-1}$	nearest integral $\mathrm{d}_{[x]+t}=$ $(bd)_{[x]+t-1}+(ad)_{[x]+t}$	... $\mathrm{f}'_{[x]+t}$ $=(\mathrm{e}+w+\mathrm{d})$	... $\mathrm{g}'_{[x]+t}$ $=(\mathrm{s}-\mathrm{f}')_{[x]+t}$
$(aw)_{[x]+t}$ $(bw)_{[x]+t}$ nearest integral $\mathrm{w}_{[x]+t}=$ $(bw)_{[x]+t-1}+(aw)_{[x]+t}$	$(ad)_{[x]+t}$ $(bd)_{[x]+t}$ nearest integral $\mathrm{d}_{[x]+t}=$ $(bd)_{[x]+t-1}+(ad)_{[x]+t}$	$(af)_{[x]+t}$ $(bf)_{[x]+t}$ ... $\mathrm{f}_{[x]+t}$ $=(\mathrm{e}+\mathrm{w}+\mathrm{d})_{[x]+t}$	$(ag)_{[x]+t}$ $(bg)_{[x]+t}$ ... $\mathrm{g}_{[x]+t}$ $=(\mathrm{s}-\mathrm{f})_{[x]+t}$
curtate* $w_{[x]+t}$ $w'_{[x]+t}$	curtate* $d_{[x]+t}$ $d'_{[x]+t}$	... $F_{[x]+t}$ $=(e^1+w+d)_{[x]+t}$ $F'_{[x]+t}$ $=(w'+d')_{[x]+t}$	... $G_{[x]+t}$ $=(s^1-F)_{[x]+t}$ $G'_{[x]+t}$ $=-(w'+d')_{[x]+t}$
curtate* $w_{[x]+t}$	curtate* $d_{[x]+t}$	... $F_{[x]+t}$ $=(e^1+w+d)_{[x]+t}$	... $G_{[x]+t}$ $=(s^1-F)_{[x]+t}$
$(aw)_{[x]+t}$ $(bw)_{[x]+t}$ nearest integral $\mathrm{w}_{[x]+t}=$ $(bw)_{[x]+t-1}+(aw)_{[x]+t}$	$(ad)_{[x]+t}$ $(bd)_{[x]+t}$ nearest integral $\mathrm{d}_{[x]+t}=$ $(bd)_{[x]+t-1}+(ad)_{[x]+t}$	$(af)_{[x]+t}$ $(bf)_{[x]+t}$ ... $\mathrm{F}_{[x]+t}$ $=(e^1+\mathrm{w}+\mathrm{d})_{[x]+t}$	$(ag)_{[x]+t}$ $(bg)_{[x]+t}$ ... $\mathrm{G}_{[x]+t}$ $=(s^1-\mathrm{F})_{[x]+t}$

"curtate" duration (t), and "fractional exposure" (1·0).

all above Methods.

Rates—

Of Mortality . . . . $q_{[x]+t}$
Of Withdrawal . . . . $(wq)_{[x]+t}$
Central Death-rate . . . . . . $m_{[x]+t}$
Central Withdrawal Rate . . . . . $(wm)_{[x]+t}$

(II). Application of Methods to the computation of the Rates experienced, and the Special Benefits granted, by Clerks' Associations.

General Characteristics of Clerks' Associations, and nature of Special Benefits Granted.

It will be convenient, at the outset, to set out as concisely as possible the leading objects and general characteristics of these Clerks' Associations, which are established in several of the cities and larger towns of England and Scotland.

The membership is restricted to *bonâ fide* clerks employed within a definite local area; and the main object of the Associations is to render assistance to the members when out of employment, or when payment of their salaries is temporarily suspended by reason of sickness.

The following is a condensed summary of the main provisions of the rules of one of these Associations established in an English city. These particulars are given by way of illustration merely, and while the main principles will not vary, the actual provisions as to age, amount, incidence of allowances, &c., may materially differ in individual Associations.

Members are admitted between the ages of 18 and 45. The rates of monthly subscriptions vary from 2*s*. to 2*s*. 9*d*. per month, under the "Low" Scale: but members can, at their option, subscribe at an increase of 50 per-cent upon the above rates (on the "Middle" Scale), or at double the minimum rates (on the "High" Scale). The benefits secured under the three scales are strictly proportionate to the increased subscriptions.

The benefits granted under the Low Scale of Subscriptions are as follow:—

(1) A weekly allowance during non-employment, or during sickness involving temporary suspension of pay, granted to members of six months' standing and upwards, of 24*s*. weekly during the first four weeks; 15*s*. weekly during the following nine weeks; and 8*s*. weekly during the remaining 13 weeks, at the end of which the allowance ceases, but is subject to renewal, for similar amounts and durations, after the member shall have held a permanent situation for at least six months. This restrictive term is reduced to three months in respect of members who have been three years upon the

books without receiving benefit: and is altogether abrogated in the case of members who have been five years upon the books without receiving benefit. There are also limitations as to the maximum aggregate amounts which may be received during the whole of membership by way of allowance, which need not be here specified in detail.

(2) An increasing death-benefit on the following scale:—To members of between six months' and five years' standing, £7. 10*s.*; to members of between five and ten years' standing, £11. 5*s.*; to members of 10 years' standing and upwards, £15.

(3) Allowances to members of not less than five years' standing, who shall be permanently unfitted for employment; and to members of not less than 15 years' standing, who shall, from old age or permanent ill-health, have no prospect of obtaining remunerative employment; also to necessitous widows and orphans of members. These allowances may be either by way of benevolent grants, or annuities, and they are at the entire discretion of the executive as to grant, amount, and duration, and are usually made from a "Special Annuity Fund" or "Reserve Fund", created and augmented by voluntary contributions, and appropriations of surplus moneys. They are not, therefore, a charge upon the members' monthly contributions.

(4) Medical attendance and medicine during sickness, the charges under this head being provided by the allocation of a fixed portion of the members' subscriptions, averaging about 4*s.* per member per annum.

(5) An "Employment Bureau", or classified register of candidates and situations, by which the members are assisted in obtaining situations when out of employment; this being equally a benefit to the individual member, and to the Association, which is thus liberated from a charge upon its funds in respect of the allowance during non-employment.

(6) A restricted allowance of death-benefit upon the Minimum Scale, or of death-benefit and medical attendance only, to members who, by removal outside the local district, or by entry into business as employers (and thus ceasing to be "clerks") have forfeited the right to the allowance

during non-employment; reduced subscriptions of 15*s.* and of 21*s.* per annum respectively being paid by such members.

The management expenses of the Association are usually defrayed (1) by an allocation of a fixed proportion (say 35 per-cent) of the members' subscriptions; (2) by incidental receipts from entry fees, donations, and miscellaneous sources of income.

Form of Card adopted, and Method of Recording the Experience.

The following is the form of card, which was, on the whole, found to be that most appropriate for recording the data:—

FRONT.

Number 4614 Scale ℒ

Age at Entry 24

Subscription £1 · 4d

Date of Entry 1 . 11 . 82

Date of Exit 1 . 4 . 91

Mode of Exit ℒ

Duration (Entry) 5 . 2

....

Duration (Exit) 8 . 4

....

Remarks..

....

....

A few explanatory notes are needed to make clear the mode of filling up the several particulars upon the cards:—

(1) *Number.*—This was filled in with the consecutive number of membership, corresponding to the policy number in an insurance office.

(2) *Scale.*—This was entered as "H" (High), "M" (Middle), or "L" (Low).

(3) *Age at Entry.*—This was, in all cases, the office age at entry, as charged, and stated in the office books. As, however, the scale of subscriptions was constant over groups of entry ages (18–30, 30–35, 35–40, and 40–45) it is probable that there was no great exactitude in stating the precise age at entry; nor does it appear

BACK.

Allowances During Sickness or Non-Employment					
Duration	Date	Weeks	Days	Amount	
3 · 5	May 1886	6	3	6	625
5 · 6	June 1888	14	=	12	300
8 · 2	Feb. 1891	3	1	3	775
·					
·					
·					
·					
·					
·					
·					

that any evidence of the true age was required or furnished. Upon the whole, and after enquiry, it seemed probable that the office age at entry was that at nearest birthday, which would be identical with Dr. Sprague's "commencing age." (*J.I.A.*, xxxi, 208.)

(4) *Subscription.*—This was filled in from the office books, the amount entered on the card being the year's subscription. In practice all subscriptions were paid at intervals of calendar months, excepting the relatively few cases of restricted benefit, where the reduced subscription of 15*s*. or 21*s*. was paid annually.

(5) *Date of Entry.*—This was entered up with the month and year in which the first subscription was paid. The subscriptions being payable in all cases on the first day of a calendar month, the date of entry always coincides with the commencement of the month recorded upon the cards.

(6) *Date of Exit.*—This was recorded, in cases of withdrawal, as the month and year in which the subscription ceased; and in cases of death, as the month and year in which death actually took place.

(7) *Mode of Exit.*—This was either by *withdrawal,* including (*a*) non-payment of subscription (*b*) removal outside the local district (*c*) entering upon business as an employer—all of which were marked "L"—or by *death,* which was marked "D."

(8) *Duration (Entry).*
(9) *Duration (Exit).* } These particulars were not entered upon the cards from the office books, but were computed and recorded later on, according to the methods specified in the first part of this paper.* It may be added that, where it is desired to deduce and tabulate the experience according to ages attained, these headings would be replaced by "Age at Entry on Observation" and "Age at Exit" respectively. The additional lines afford opportunity for similar entries at subsequent investigations.

(10) *Remarks.*—Here particulars were entered of any special incidents of the case, including changes of scale during membership; revival after lapse; transfer to restricted scale of benefit, by removal or entry upon business; transfers to Annuity or Benevolent Funds; &c.

* See pp. 5, 6.

(11) *Allowances during Sickness or Non-Employment.*—The entries under this head being sometimes numerous, and increasing with the duration of membership, it was found preferable to preserve the back of the cards for these particulars. The first column "duration" (representing the duration of actual membership at the date when allowance commenced) was not supplied from the office books, but computed and recorded later. The "date" was that of the month and year when benefit commenced; the "weeks" the number of weeks (and days) during which the allowance was continuously paid; and the "amount" the aggregate sum received in respect of each such series of continuous payments.

Methods adopted in deducing Rates of Mortality, Withdrawal, and Allowance during Non-Employment.

The method actually selected for investigating the particular experience here under review was that of Mean Durations, as this method offered great facilities in the actual computations, and undoubtedly gave very accurate results in this particular case.

The processes of deducing the numbers exposed to risk, and the rates of mortality and withdrawal, in central ages at entry and individual years of duration, have been fully explained in the first part of this paper, the tabulation being in the form set out in Schedule (C), on pages 16 and 17.

The *rate of mortality* was experimentally computed in this form; but it was seen from the outset that the aggregate number of deaths (128) was quite too few to give trustworthy, or indeed, workable, results; and it was ultimately decided to treat the rate of mortality as a function of the age only, irrespective of duration of membership. The age attained was assumed to be equal to the sum of the office age at entry (x) and the curtate duration (t); and all cases having an (assumed) age of $(x+t)$ were brought together so that the numbers exposed to risk $(\bar{E}_{x+t})$ and the deaths (d_{x+t}) could be compared, and the rate of mortality computed. The rate thus deduced was equal to the central death-rate (m_{x+t}), and the values of (q_{x+t}) were deduced by the formula of relation

$$q_{x+t} = \frac{2m_{x+t}}{2+m_{x+t}}$$

It will be seen that, as the cases had been grouped in central ages at entry, and years of duration, the effect of this was that the values of m_{x+t} (and consequently those of q_{x+t}) were deduced from the numbers exposed to risk, and the deaths at five grouped ages $(x+t-2)$, $(x+t-1)$, $(x+t)$, $(x+t+1)$, and $(x+t+2)$. This was in effect a first graduation by summation in fives. The results thus deduced were combined with the experience of the previous seven years 1880–7, and a graduated rate was ultimately adopted, which represented approximately the experience of the 12 years 1880–1892.

The rate of mortality thus adopted, as a basis for money-values, is set out in Table I.* The actual experience did not extend beyond age 60, and at this age a junction was conveniently effected with a table of mortality representing the experience of males in city districts employed in similar occupations to those of the members whose experience was under investigation.

As regards the *rate of withdrawal,* the experience was much more extensive, there having been 1,222 cases of withdrawal during the period of observation. The actual rate of withdrawal was deduced for each central age at entry and each year of duration by the formula

$$(wm)_{[x]+t} = \frac{w_{[x]+t}}{\mathrm{E}_{[x]+t}}$$

the values of $(wq)_{[x]+t}$ being then computed by the formula of relation

$$(wq)_{[x]+t} = \frac{2(wm)_{[x]+t}}{2+(wm)_{[x]+t}}$$

The resulting values are set out in Table II for each central age at entry and each year of duration. It will be seen that the rate of withdrawal (at all entry ages) averages about 16 per-cent in the first year of duration, about 9 per-cent in the fifth year, 5 per-cent in the tenth year, 4 per-cent in the fifteenth year, and thence diminishes steadily to practical extinction at about the thirtieth year. It is also to be noted that the rates are somewhat materially lower at central age at entry 20 (especially in the earlier years of duration) than the rates at higher entry ages in corresponding years of duration. As there was a considerable preponderance of cases at the earlier entry ages and durations, this diminution in the rate of withdrawal was somewhat noteworthy, and could hardly be disregarded in the computation of money-values.

* For Tables I to VIII, see pages 78–84.

A rate of withdrawal was ultimately adopted, which, while having regard to the results set out in Table II, was well within that actually experienced, and thus left a fair margin for contingencies in this respect.

As regards the *rate of non-employment*, the method adopted was as follows: The dates of the several allowances, the number of weeks and days during which they were continuously paid, and the aggregate amounts paid in respect of each continuous allowance, having been recorded on the backs of the cards, as already explained, the duration of membership as at the date of each such allowance was computed to one decimal place, and entered in the column headed "Duration." The cards were then sorted (1) according to central ages at entry; (2) according to year of duration when entering on benefit; and the cases counted and tabulated, so as to show, for each central age at entry and each successive year of duration, the number of cases of benefit, the total number of weeks during which benefit was actually paid, and the aggregate amount paid in each case.

The allowance was, as has been said, a variable one, diminishing after 4 weeks', and again after 13 weeks', continuous pay, and ceasing after 26 weeks of total pay. In order to give effect to this, in a form convenient for computation, a further column was added, giving the number of weeks over which the allowance would have extended, if the amount of the weekly pay had been, throughout, the maximum amount which was in fact payable during the first four weeks only. This was readily arrived at, by dividing the aggregate amount received, by the maximum weekly allowance, which, of course, varied according to the scale of membership and subscription (High, Middle, or Low); and could be computed in each individual case, or in each group of cases where the central entry age, the year of duration, and the scale of membership, were all identical.

The number of weeks thus deduced, which I have termed "equivalent weeks", numbered 2,776 in the aggregate, as compared with 4,206 weeks during which the reducible allowance was actually paid; and the effect of this was to reduce the tabulated number of weeks in the average ratio of about 3 to 2.

By the adoption of this device, a good deal of labour was saved in the following stages. It proceeds, as will have been seen, upon the reasonable assumption that the value of an allowance extending, for example, over fifteen weeks, of £1. 4*s*. during the first four weeks, of 15*s*. during the following nine weeks,

and of 8*s*. for the remaining two weeks—or £12. 7*s*. in all—cannot differ materially from the value of an allowance at the constant rate of £1. 4*s*. weekly extending over 10·3 $\left(=\frac{12{\cdot}35}{1{\cdot}2}\right)$ weeks, and also amounting, in the aggregate to £12. 7*s*.

Upon the basis of the number of "equivalent weeks" thus deduced, the average number of weeks' allowance per member was deduced (1) as a function of the year of duration only; (2) as a function of the age attained only; the formulæ being respectively

$$u_t = \frac{v_t}{\bar{E}_t}$$

and
$$u_{x+t} = \frac{v_{x+t}}{\bar{E}_{x+t}}$$

where v_t and v_{x+t} represent the aggregate number of "equivalent weeks" arising in the $(t+1)$th year of duration, and in the year following age $(x+t)$, respectively; $\bar{E}_t$ and $\bar{E}_{x+t}$ represent the numbers exposed to risk in the $(t+1)$th year of duration, and in the year following age $(x+t)$ respectively; and u_t and u_{x+t} represent the resulting rates of non-employment. In deducing these rates, as at ages attained, a special correction was made, in respect of the non-payment of benefit during the first six months of membership.

In Table III, I have set out the rate of non-employment computed upon these two bases. As regards the rate experienced in years of duration, it might be anticipated that there would be some indications of a selection against the Association in the early years of duration, arising from members entering who were in expectation of benefitting by the allowances in this respect at an early date. If the rates, as tabulated in years of duration, are closely examined, as set out in column (4) of Table III, regard being given to the fact that the rate during the first year should practically be doubled for purposes of comparison, it will be seen that there are indications, in the early years of duration, of the effects of some such causes: but these indications cannot be said to be strongly marked.

As regards the tabulation of the rates according to ages attained, it will be seen from column (2) that the rate tends to diminish from age 20 to about age 31, after which it tends on the whole to increase with the age, up to age 64, the highest age under observation.

It was, upon the whole, thought to be preferable to deduce the money values upon the basis of the rate of non-employment as computed at ages attained. As this function is one of some interest, I have appended in Table IV the rate of non-employment as experienced in the several periods 1862-79, column (2); 1880-87, column (3); and 1888-92, column (4), at ages attained; and also, in column (5), the rates adopted for the computation of money values, after consideration of the whole experience thus available.

The rates above age 64 not being obtainable, an assumed rate of allowance was tentatively adopted; and, having regard to the restrictions under the rules as to the duration of allowance, and the conditions under which the allowance could be resumed, as well as those limiting the aggregate amount receivable by a member, a rate increasing up to a maximum of two weeks' allowance per annum per member, was deemed to be fully sufficient to meet the case.

Methods of Computing Values of Benefits, and Valuation Tables and Results.

Upon the bases specified above, as to the rates of mortality, of withdrawal, and of allowance during non-employment, there were computed, for each central age at entry, and for each successive year of duration, the value (1) of annuities payable throughout membership (2) of the varying assurance at death and (3) of the benefit payable during non-employment; interest being taken throughout at a rate of 3 per-cent per annum.

(1) The values of the annuity payable throughout membership, that is, until cessation by death or withdrawal, were computed upon the basis of the approximate formula (See Appendix E)

$$q''_{[x]+t} = q_{[x]+t} + (wq)_{[x]+t} - [q \times (wq)]_{[x]+t}$$

where q'' represents the probability of either death or withdrawal during the year of duration; whence we have

$$p''_{[x]+t} = 1 - q''_{[x]+t}$$

and $$\log vp''_{[x]+t} = \Delta \log D''_{[x]+t}$$

whence the values of $N''_{[x]+t}$ and of $a''_{[x]+t}$ can be deduced by the ordinary processes.

(2) The values of the assurance, payable at death, with allowance for mortality and withdrawal during life, were based upon the approximate formula (See Appendix E)

$$q'_{[x]+t} = q_{[x]+t} - \tfrac{1}{2}[q \times (wq)]_{[x]+t}$$

where q' represents the probability of death, allowing for withdrawals: and we have

$$q'_{[x]+t} \times D''_{[x]+t} \times v = d'_{[x]+t} v^{x+t+1} = C'_{[x]+t}$$

or

$$\log q'_{[x]+t} + \log D''_{[x]+t} + \log v = \log C'_{[x]+t}$$

whence the values of $M'_{[x]+t}$, of $A'_{[x]+t}$, and of the varying assurance, can readily be deduced in the usual way.

(3) The values of the allowance during non-employment were obtained by the formula

$$U'_{[x]+t} = \frac{\Sigma(D''_{[x]+t+\frac{1}{2}} \cdot u_{[x]+t})}{D''_{[x]+t}}$$

where $U'_{[x]+t}$ represents the value of a constant benefit of one per week during non-employment, with allowance for mortality and withdrawals, and

$$\Sigma(D''_{[x]+t+\frac{1}{2}} \cdot u_{[x]+t}) = D''_{[x]+t+\frac{1}{2}} \cdot u_{[x]+t} + D''_{[x]+t+1\frac{1}{2}} \cdot u_{[x]+t+1} + \ldots$$

The tabular values, computed as above, represent the values of the several benefits to members entering at office age $[x]$, who were in existence (at the date of valuation) at the precise durations 0, 1, 2, 3, . . . t years. By taking the means of these values throughout, factors were obtained appropriate for the valuation of the benefits or contributions of members whose curtate durations were, at the date of valuation 0, 1, 2, 3, . . . t years, and whose fractional exposures were, on the average, half-a-year.

The cards representing the cases "existing" at the date of the valuation having been already sorted according to central ages at entry and curtate durations, the mean valuation factors, arrived at as above, were then applied to the valuation of the cases in each year of duration, regard being also given to the scale under which the members had severally subscribed. By this means the liability in respect of the death-benefit, and of the allowance during non-employment, was ascertained; and the values of the members' annual subscriptions were similarly computed by the employment of mean annuity-values.

In computing the values of the subscriptions, allowance was made for the non-payment of the monthly subscriptions by

members whilst in receipt of benefit. This was arrived at by increasing the amount of the maximum weekly allowance during non-employment, by the amount of the average *weekly* subscription of the members (in each of the three scales) as a whole, increased in the proportion of actual to "equivalent" weeks of benefit.

The value of the estimated net premiums was arrived at by deducting from the value of the office subscriptions payable (1) the percentage available under the rules for management expenses, (2) the value of the constant deduction in respect of medical attendance, &c. Finally, by deducting the aggregate value of the estimated net premium from the aggregate values of the death benefit and the allowance during non-employment, and carefully eliminating negative values, the amount of the estimated liability was arrived at.

As to the effect of an estimated allowance for Withdrawals upon the Net Premiums and Valuation Factors computed, and upon the Reserves or Net Liability.

The effect of making an allowance for secessions or withdrawals, in the valuation of Benefit Societies, is a subject that does not appear to have received much attention in our published transactions; although it must, as I imagine, come frequently under the attention of the actuary. It is quite impossible for me, within the limits of the present paper, to discuss this interesting question at any length, or at all adequately; but I have thought that it would be of interest to append some comparative results, as to the effect of making an allowance for withdrawals, in the case of the particular benefits here dealt with (1) on the amounts of the net premium and valuation factors, (2) on the amount of the reserves, or net liability.

I have assumed, for the sake of simplicity, that the benefit at death is a constant amount of £10; that the allowance during non-employment is at the constant rate of £1 per week, without reduction, but limited as to incidence and duration according to the conditions specified on pages 164 and 165; and I have adopted the rate of mortality as specified in Table I; a graduated rate of withdrawal based upon the rates shown in Table II, and ceasing at the expiration of 30 years' duration; and a rate of non-employment as shown in Table IV, column (5); with interest throughout at 3 per-cent.

I have preferred, in these illustrative examples, to assume a rate of withdrawal which, upon the whole, is fully equal to that actually experienced; it will, however, be understood that in the practical valuation of such an Association, the rate of withdrawal assumed as likely to operate in the future should be materially below that actually obtaining in the immediate past.

I have computed, and give in Table V, the valuation factors $\bar{A}'_{[x]+t}$, $U'_{[x]+t}$, and $\bar{a}''_{[x]+t}$, with allowance for withdrawals; also the factors $\bar{A}_{[x]+t}$, $U_{[x]+t}$, and $\bar{a}_{[x]+t}$, without allowance for withdrawals; taking values of $[x]=20$, 30, and 40; and of $t=0$, 1, 3, 5, 10, 15, 20, 25, and 30. I have also computed the value of $\pi'_{[x]}$, the net premium, with allowance for withdrawals, required to provide £10 at death, and £1 weekly during non-employment; and of $\pi_{[x]}$ the net premium, without allowance for withdrawals, computed to provide the same benefits. These valuation factors and net premiums are, as might have been expected, materially reduced where the element of withdrawal is introduced, and especially in the early years of duration.

In Table VI are given what may be termed the true net premium reserve values at the ages at entry, and after the several durations, above mentioned. Here the reserve values in columns (2), (3), and (4) are computed on the basis of a net premium and valuation factors allowing for withdrawals; while in columns (5), (6), and (7) the reserves are upon the basis of a net premium and of valuation factors without allowance for withdrawals. These reserve values will be built up respectively by the accumulations of the net premiums as computed, assuming, of course, that the rates of mortality, of withdrawal, and of non-employment do not differ from those assumed.

It will be seen that in these examples the reserve values are, in the early years of duration, less when the element of withdrawal is introduced; but that after between 10 and 15 years' duration, the introduction of the withdrawals increases the reserve values throughout. It does not of course follow, that these relations would always hold good, and the comparative results would probably be modified, according to the rate of withdrawal assumed in successive years of duration, and the progression of that rate. It is, however, clear, where (as in the present examples) the rate of withdrawal is assumed to operate over a fixed term of years only, that the reserve values *after the expiration of that term* must throughout be greater where the element of withdrawal enters into the net premium: for the other valuation factors (the effect

of withdrawal having ceased to operate) are now identical, and the lower net premium will necessarily produce a greater reserve value.

In the above examples it is assumed that the Actuary has a free hand in the selection and computation of his valuation factors, and that the office scale of subscriptions will, after a reasonable allowance for expenses and other charges, be sufficient to provide the larger net premium required where no allowance is made for withdrawals. As, however, the expenses of management and other charges will usually absorb a stated proportion of the office subscriptions, the more usual case will be that where the *available net premium* applicable to benefits is represented by the fixed proportion of the subscriptions remaining.

Let us first take the case where this available benefit premium is found to be just sufficient to provide the risks without any allowance for withdrawals. I have computed in Table VII the comparative reserve values upon this basis, with and without allowance for withdrawals; but always upon the assumption that the net premium valued is that available as above, and which will provide for the benefits, assuming that there are no withdrawals.

Taking the columns (2), (3), and (4), and comparing the results with those set out in columns (5), (6), (7), it will be seen that the element of withdrawal introduces (upon this basis) negative values in the early years of duration, and that the reserve values are throughout diminished, where allowance is made for withdrawals. After 30 years' duration, however, when the effect of withdrawals ceases, the reserve values will be identical under both assumptions; as the net premium employed in this case is throughout the same in the two cases.

It may be added that the reserve values set out in columns (5), (6), and (7) will be throughout built up (without allowance for withdrawals) precisely by the net premiums assumed; and that the reserve values set out in columns (2), (3), and (4) will consequently be materially less than those that would have been built up (with allowance for withdrawals) by the assumed net premiums; the difference being represented by the anticipated value of the future profit arising from withdrawals.

Taking now the case where the available net premium, after providing for expenses and charges, is only just sufficient to provide the benefits granted, with full allowance for withdrawals, we have the results set out in Table VIII. Here the element

of withdrawal materially reduces the reserve values in columns (2), (3), and (4), which are throughout less than those in columns (5), (6), and (7), until the effects of withdrawals ceases, when they are identical with those deduced without allowance for withdrawals.

The values in columns (2), (3), and (4) are here precisely built up (with allowance for withdrawals) by the net premiums assumed; but these premiums would be quite insufficient to build up the greater reserve values (without allowance for withdrawals) set out in columns (5), (6), and (7); the difference representing the present value of the loss which would arise if there should in the future be no withdrawals.

In the case of the Association whose experience has been under review in the earlier portion of this paper, it was found that the available office subscriptions, after allowing for expenses and medical charges, was somewhat more than sufficient to provide the benefits granted, upon the assumption that the future rate of withdrawal would be reasonably below that actually experienced. It will probably be found, in many cases, that there is a very narrow margin in this respect; and that, therefore, the premium available for benefits would not be sufficient to provide reserve values computed without allowance for withdrawals. The rate of withdrawal likely to obtain in the future history of the Association thus becomes a most important element in the case; as, if from any cause the rate experienced falls materially below that assumed in the valuation, the net premiums available will no longer be sufficient to provide the necessary reserves.

It therefore behoves the Actuary to exercise great caution in his assumptions as to the rate of withdrawal. If, on the one hand, he altogether ignore this element, or adopt a rate materially below that indicated by the experience, he will probably bring out an immediate and large (but at the same time somewhat illusory) deficiency in the funds; while if, on the other hand, he assume a rate of withdrawal practically identical with that actually experienced in the past, he runs the risk that the rate obtaining in the future may be materially below his estimates, and that a grave deficiency may thus arise at future valuations. In practice, a careful judgment must be exercised, as to the sufficiency, upon reasonable assumptions as to the rate of withdrawal, of the available net premium to provide for the risks; and the basis upon which to proceed in the computations of the reserves, must be largely determined by a careful analysis of the

circumstances and experience of the particular Association, and its general financial position.

As an illustration of the important financial considerations involved in these questions, I have roughly computed the aggregate amount of reserves which would have been required, upon the different assumptions made in Tables VI, VII, and VIII, in respect of the 2,881 members actually existing, as at 31 December 1892, in the Association whose experience I have dealt with in this paper. I have taken, throughout, supposititious benefits of £10 at death, and of £1 weekly during non-employment. The reserve values were computed, at quinquennial ages and durations, by means of the values given in Tables VI, VII, and VIII, so as to give the aggregate reserves (1) with allowance for withdrawals, (*a*) in both valuation factors and net premium, (*b*) in valuation factors only; (2) without allowance for withdrawals, (*a*) in both valuation factors and net premium, (*b*) in valuation factors only. The application of these four cases to the "model office" necessarily involves the assumption that the available benefit premiums are, throughout, sufficient to cover the net premiums computed without allowance for withdrawals. The resulting Aggregate Reserve Values are as follow:—

Aggregate Reserve Values.

Basis of Reserves	Basis of Net Premiums	Aggregate Reserves	Ratios per-cent	Table
With Allowance for Withdrawals	With	£8,856	100·0	VI (*a*)
Without „ „	Without	8,901	100·5	VI (*b*)
With „ „	Without	5,327	60·2	VII (*a*)
Without „ „	With	14,156	159·9	VIII (*b*)

Table I.—Clerks' Association.

Graduated Rate of Mortality, as employed in computation of money values, based upon actual experience, 1880–1892.

Age $(x+t)$	Rate of Mortality q_{x+t}
(1)	(2)
20	·0046
21	·0047
22	·0048
23	·0049
24	·0052
25	·0054
26	·0057
27	·0059
28	·0062
29	·0065
30	·0068
31	·0071
32	·0073
33	·0076
34	·0079
35	·0082
36	·0085
37	·0088
38	·0092
39	·0096
40	·0101
41	·0106
42	·0112
43	·0119
44	·0126
45	·0133
46	·0142
47	·0150
48	·0159
49	·0167
50	·0176
51	·0186
52	·0197
53	·0211
54	·0230
55	·0255
56	·0285
57	·0321
58	·0366
59	·0402

Table II.—Clerks' Association.

Rates of Withdrawal, as actually experienced (1888–92) *scheduled according to Central Ages at Entry and Years of Duration.*

Year of Duration t	Central Age at Entry $[x]$ — 20	25	30	35	40	45	All Entry Ages	Year of Duration t
	Rate of Withdrawal $100(wq_{[x]+t})$							
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0	11·4	19·7	17·2	18·4	17·8	22·1	16·7	0
1	10·9	13·1	10·2	18·4	23·4	...	12·0	1
2	9·8	11·9	13·1	14·5	9·0	...	10·9	2
3	9·2	14·9	13·9	5·9	14·5	18·9	12·2	3
4	11·1	9·6	7·2	1·3	10·5	...	9·4	4
5	9·2	9·2	9·7	9·8	9·8	...	9·7	5
6	7·2	8·6	6·8	9·2	6·1	10·5	8·0	6
7	5·2	6·2	3·7	7·3	7·1	...	5·5	7
8	6·4	6·3	4·7	4·7	12·1	8·7	6·1	8
9	4·1	3·7	4·3	3·2	4·3	...	3·8	9
10	5·3	5·1	4·7	3·6	...	...	4·8	10
11	6·0	5·6	5·3	6·3	...	...	5·8	11
12	3·9	7·9	2·6	7·3	...	...	5·2	12
13	5·9	2·2	6·4	3·0	...	...	4·3	13
14	6·8	3·9	4·0	3·9	20·0	...	5·2	14
15	1·8	5·6	2·8	5·2	...	...	3·7	15
16	6·2	3·5	7·2	...	...	...	4·7	16
17	4·6	...	4·1	...	...	...	2·5	17
18	4·7	2·5	4·8	...	...	...	3·7	18
19	1·9	3·4	...	13·3	...	...	2·8	19
20	6·4	4·4	...	...	...	...	4·6	20
21	2·3	4·6	...	..	...	...	3·8	21
22	...	...	...	...	...	...	...	22
23	2·0	...	...	...	...	...	1·0	23
24	...	...	...	...	...	...	...	24
25	1·7	2·6	...	7·4	...	...	2·2	25
26	2·0	5·0	3·5	...	...	...	3·0	26
27	...	...	4·3	...	...	...	0·9	27
28	4·4	...	...	...	...	...	1·4	28
29	...	6·7	...	...	...	...	2·3	29

TABLE III.—CLERKS' ASSOCIATION.

Rates of Non-Employment, as actually experienced (1888–92), *scheduled* (1) *according to ages attained* (2) *according to years of duration.*

Age attained $x+t$	Rate of Non-Employment u_{x+t}	Year of Duration t	Rate of Non-Employment u_t
(1)	(2)	(3)	(4)
20	·106	0	·087
21	·262	1	·245
22	·174	2	·122
23	·236	3	·204
24	·283	4	·311
25	·139	5	·159
26	·219	6	·167
27	·175	7	·248
28	·143	8	·165
29	·151	9	·100
30	·216	10	·185
31	·127	11	·222
32	·167	12	·247
33	·141	13	·113
34	·165	14	·152
35	·151	15	·139
36	·278	16	·189
37	·217	17	·304
38	·195	18	·349
39	·283	19	·177
40	·125	20	·012
41	·109	21	·290
42	·143	22	·061
43	·196	23	...
44	·171	24	·013
45	·139	25	·206
46	·341	26	·158
47	·509	27	·508
48	·360	28	·383
49	·117	29	·864
50	·269	...	...
51	·066	...	...
52	·709	...	...
53	...	...	...
54	·255	...	...
55	·141	...	...
56	·660	...	...
57	·535	...	...
58	...	...	...
59	...	...	...
60	·680	...	...
61	...	...	...
62	...	...	...
63	·300	...	...
64	2·787	...	...

TABLE IV.—CLERKS' ASSOCIATION.

Graduated Rates of Non-Employment, based upon the Experience of successive periods 1862–1879, 1880–87, *and* 1888–92; *also the rate adopted for computation of Money Values.*

Age attained	Period of Experience			Rate employed in computation of Money Values	Age attained
	1862–1879	1880–1887	1888–1892		
(t)	Rate of Non-Employment (u_{x+t})				(t)
(1)	(2)	(3)	(4)	(5)	(6)
20	·266	·259	·214	·252	20
21	·268	·256	·213	·250	21
22	·268	·258	·212	·249	22
23	·263	·260	·211	·249	23
24	·250	·264	·209	·249	24
25	·243	·269	·199	·248	25
26	·232	·270	·192	·246	26
27	·226	·269	·182	·242	27
28	·228	·266	·172	·237	28
29	·232	·259	·166	·231	29
30	·238	·248	·165	·226	30
31	·247	·241	·160	·219	31
32	·256	·231	·163	·214	32
33	·260	·224	·169	·211	33
34	·260	·216	·177	·210	34
35	·256	·217	·189	·213	35
36	·261	·223	·203	·220	36
37	·265	·239	·204	·229	37
38	·269	·253	·200	·238	38
39	·270	·277	·194	·244	39
40	·281	·288	·179	·249	40
41	·291	·292	·166	·250	41
42	·308	·289	·168	·252	42
43	·341	·284	·188	·256	43
44	·385	·280	·215	·261	44
45	·421	·277	·244	·268	45
46	·435	·275	·277	·276	46
47	·446	·274	·290	·284	47
48	·433	·269	·297	·293	48
49	·448	·252	·283	·303	49
50	·408	·249	·276	·316	50
51	...	·274	·259	·335	51
52	...	·344	·277	·358	52
53	...	·440	·280	·384	53
54	...	·585	·297	·416	54
55	...	·719	·298	·454	55
56	...	·820	·322	·499	56
57	...	·841	·305	·538	57
58	...	·845	·292	·577	58
59	...	·774	·271	·616	59
60	...	·733	·303	·655	60
61	...	...	·295	·710	61
62	...	...	·352	·780	62
63	...	...	·496	·870	63
64	...	...	·682	·980	64

Table V.—Clerks' Association.

Values of Assurances at Death, of Allowance during Non-Employment, of Annuities, and of Net Premiums, based upon the Rate of Mortality as shown in Table I; *a Graduated Rate of Withdrawal (where assumed), based upon Table* II; *and the Rate of Non-Employment as shown in Table* IV *column* (5), *with Interest at* 3 *per-cent. The Net Premiums are computed (with and without Withdrawals) to provide Benefits of* £10 *at Death, and of* £1 *weekly during Non-Employment.*

Duration	With Allowance for Withdrawals: Value of Assurance of 1 at Death	Value of Allowance of 1 Weekly during Non-Employment	Value of Annuity of 1 per annum	Without Allowance for Withdrawals: Value of Assurance of 1 at Death	Value of Allowance of 1 Weekly during Non-Employment	Value of Annuity of 1 per annum	Duration
(t)	$A'_{[x]+t}$	$U'_{[x]+t}$	$\bar{a}''_{[x]+t}$	$A_{[x]+t}$	$U_{[x]+t}$	$\bar{a}_{[x]+t}$	(t)
	$\pi'_{[20]}=\cdot397$		Age at Entry $[x]=20$			$\pi_{[20]}=\cdot476$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	·105	2·65	9·33	·337	7·31	22·45	0
1	·123	3·08	10·25	·343	7·44	22·22	1
3	·156	3·64	11·68	·358	7·45	21·72	3
5	·189	4·09	12·68	·373	7·47	21·21	5
10	·267	5·06	14·12	·413	7·60	19·86	10
15	·355	6·26	15·01	·457	7·95	18·39	15
20	·442	7·36	14·98	·505	8·38	16·77	20
25	·528	8·41	14·31	·557	8·87	15·01	25
30	·611	9·50	13·16	·611	9·50	13·16	30
	$\pi'_{[30]}=\cdot456$		Age at Entry $[x]=30$			$\pi_{[30]}=\cdot585$	
0	·117	2·30	7·61	·413	7·60	19·86	0
1	·144	2·85	8·72	·422	7·65	19·57	1
3	·192	3·62	10·27	·439	7·78	18·99	3
5	·238	4·33	11·36	·457	7·95	18·39	5
10	·332	5·63	12·21	·505	8·38	16·77	10
15	·437	7·03	12·44	·557	8·87	15·01	15
20	·538	8·38	11·85	·611	9·50	13·16	20
25	·636	9·65	10·69	·669	10·13	11·19	25
30	·723	10·72	9·39	·723	10·72	9·39	30
	$\pi'_{[40]}=\cdot606$		Age at Entry $[x]=40$			$\pi_{[40]}=\cdot794$	
0	·154	2·62	6·86	·505	8·38	16·77	0
1	·189	3·26	7·78	·515	8·46	16·42	1
3	·250	4·17	8·97	·535	8·65	15·73	3
5	·307	5·00	9·73	·557	8·87	15·01	5
10	·420	6·60	9·92	·611	9·50	13·16	10
15	·545	8·26	9·48	·669	10·13	11·19	15
20	·651	9·62	8·55	·723	10·72	9·39	20
25	...	...	...	...	...	...	25
30	...	...	...	...	...	...	30

Table VI.—Clerks' Association.

Comparison of Reserve Values for Benefits of £10 *at death, and of* £1 *weekly during Non-Employment; computed upon the basis of the factors given in Table* V, *respectively with and without allowance for Withdrawals.*

Duration	(*a*) With Allowance for Withdrawals			(*b*) Without Allowance for Withdrawals			Duration
	Age at Entry			Age at Entry			
(*t*)	20	30	40	20	30	40	(*t*)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	·24	·31	·43	·29	·42	·57	1
3	·56	·86	1·23	·69	1·06	1·51	3
5	·95	1·53	2·17	1·11	1·76	2·52	5
10	2·12	3·38	4·79	2·28	3·62	5·16	10
15	3·85	5·73	7·96	3·77	5·66	7·94	15
20	5·83	8·36	10·95	5·45	7·91	10·49	20
25	8·01	11·13	...	7·29	10·27	...	25
30	10·38	13·67	...	9·34	12·46	...	30
$\pi_{[x]}$ (as valued)	·397	·456	·606	·476	·585	·794	$\pi_{[x]}$ (as valued)

Table VII.—Clerks' Association.

Comparison of Reserve Values for Benefits of £10 *at death, and of* £1 *weekly during Non-Employment; computed upon the basis of the factors given in Table* V; the net premiums employed being in all cases those computed WITHOUT allowance for Withdrawals.

Duration	(*a*) With Allowance for Withdrawals			(*b*) Without Allowance for Withdrawals			Duration
	Age at Entry			Age at Entry			
(*t*)	20	30	40	20	30	40	(*t*)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	(− ·57)	(− ·81)	(− 1·03)	·29	·42	·57	1
3	(− ·36)	(− ·48)	(− ·45)	·69	1·06	1·51	3
5	(− ·06)	·05	·34	1·11	1·76	2·52	5
10	1·01	1·81	2·92	2·28	3·62	5·16	10
15	2·66	4·15	6·18	3·77	5·66	7·94	15
20	4·65	6·80	9·34	5·45	7·91	10·49	20
25	6·88	9·75	...	7·29	10·27	...	25
30	9·34	12·46	...	9·34	12·46	...	30
$\pi_{[x]}$ (as valued)	·476	·585	·794	·476	·585	·794	$\pi_{[x]}$ (as valued)

TABLE VIII.—CLERKS' ASSOCIATION.

Comparison of Reserve Values for Benefits of £10 at death and of £1 weekly during Non-Employment; computed upon the basis of the factors given in Table V; the net premiums employed being in all cases those computed WITH allowance for Withdrawals.

Duration	(a) With Allowance for Withdrawals			(b) Without Allowance for Withdrawals			Duration
	Age at Entry			Age at Entry			
(t)	20	30	40	20	30	40	(t)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	·24	·31	·43	2·05	2·94	3·66	1
3	·56	·86	1·23	2·41	3·51	4·47	3
5	·95	1·53	2·17	2·78	4·13	5·34	5
10	2·12	3·38	4·79	3·84	5·78	7·63	10
15	3·85	5·73	7·96	5·22	7·59	10·04	15
20	5·83	8·36	10·95	6·77	9·61	12·26	20
25	8·01	11·13	...	8·48	11·72		25
30	10·38	13·67	..	10·38	13·67	...	30
$\pi_{[x]}$ (as valued)	·397	·456	·606	·397	·456	·606	$\pi_{[x]}$ (as valued)

APPENDIX (A).

PERIOD OF OBSERVATION LIMITED BY CALENDAR YEARS. EXACT DURATION FORMULÆ. (SCHEDULE B.)

WE have, for the number exposed to risk at age $[x]$,

$$\begin{aligned}\bar{E}_{[x]} &= n_{[x]} + (s-s')_{[x]} - (e+w+d)_{[x]} + (e'+w'+d')_{[x]} \\ &= n_{[x]} + (s-f)_{[x]} - (s'-f')_{[x]} \\ &= n_{[x]} + g_{[x]} - g'_{[x]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)\end{aligned}$$

also for the number exposed at age $[x]+1$

$$\begin{aligned}\bar{E}_{[x]+1} &= \bar{E}_{[x]} + g'_{[x]} + g_{[x]+1} - g'_{[x]+1} \\ &= \bar{E}_{[x]} + g_{[x]+1} - \Delta g'_{[x]} \quad . \quad . \quad . \quad . \quad (2)\end{aligned}$$

and, generally $\bar{E}_{[x]+t} = \bar{E}_{[x]+t-1} + g_{[x]+t} - \Delta g'_{[x]+t-1}$ . . . (3)

If now we insert in formula (2) the value of $\bar{E}_{[x]}$ as given in formula (1), we have:

$$\begin{aligned}\bar{E}_{[x]+1} &= n_{[x]} + g_{[x]} + g_{[x]+1} - g'_{[x]} - \Delta g'_{[x]} \\ &= n_{[x]} + g_{[x]} + g_{[x]+1} - g'_{[x]+1} \\ &= n_{[x]} + \Sigma_0^1(g) - g'_{[x]+1}\end{aligned}$$

and, generally $\bar{E}_{[x]+t} = n_{[x]} + \Sigma_0^t(g) - g'_{[x]+t}$ . . . . (4)

where $\Sigma_0^t(g) = (g_{[x]} + g_{[x]+1} + \ldots\ldots + g_{[x]+t})$

Mean Duration Formulæ. (Schedule C.)

Here $$s'=\frac{s}{2},\ e'=\frac{e}{2},\ w'=\frac{w}{2},\ d'=\frac{d}{2};$$

also, $$f'=e'+w'+d'=\frac{e+w+d}{2}=\frac{f}{2}$$

and $$g'=s'-f'=\frac{s-f}{2}=\frac{g}{2}.$$

Inserting these mean values in formulæ (1) (3) and (4) respectively, we have:

$$\bar{E}_{[x]}=n_{[x]}+\frac{g_{[x]}}{2}\quad\ldots\ldots\ldots\ldots\quad(5)$$

$$\bar{E}_{[x]+t}=\bar{E}_{[x]+t-1}+\frac{g_{[x]+t}+g_{[x]+t-1}}{2}\quad\ldots\quad(6)$$

$$\bar{E}_{[x]+t}=n_{[x]}+\Sigma_0^t(g)-\frac{g_{[x]+t}}{2}\quad\ldots\ldots\quad(7)$$

Nearest Duration Formulæ, as applied (in Schedule D) for deducing *either* the Rate of Mortality or that of Withdrawal.

(*a*) Rate of Mortality.

Let s, e and w, represent the numbers surviving, existing and withdrawing, as tabulated by this method; so that we have, for initial values:

$$\mathrm{s}_{[x]}=(as)_{[x]}\qquad \mathrm{e}_{[x]}=(ae)_{[x]}\qquad \mathrm{w}_{[x]}=(aw)_{[x]}$$

and for subsequent values,

$$\mathrm{s}_{[x]+t}=[(bs)_{[x]+t-1}+(as)_{[x]+t}]$$

$$\mathrm{e}_{[x]+t}=[(be)_{[x]+t-1}+(ae)_{[x]+t}]$$

$$\mathrm{w}_{[x]+t}=[(bw)_{[x]+t-1}+(aw)_{[x]+t}]$$

the symbols a and b, representing throughout the reference to the beginning and the end respectively of the year of duration.

Also let $$\mathbf{f}_{[x]+t}=(\mathrm{e}+\mathrm{w})_{[x]+t}+d_{[x]+t-1}$$

where $d_{[x]+t-1}$ represents, as usual, the deaths actually occurring in the tth year of duration,

and let $$\mathbf{g}_{[x]+t}=(\mathrm{s}-\mathbf{f})_{[x]+t}.$$

Then we have for the numbers exposed to the risk of death,

$$\mathrm{E}_{[x]}=n_{[x]}+\mathbf{g}_{[x]}\quad\ldots\ldots\ldots\quad(a)$$

$$\mathrm{E}_{[x]+1}=\mathrm{E}_{[x]}+\mathbf{g}_{[x]+1}\quad\ldots\ldots\quad(b)$$

and, generally, $$\mathrm{E}_{[x]+t}=\mathrm{E}_{[x]+t-1}+\mathbf{g}_{[x]+t}\quad\ldots\ldots\quad(c)$$

Inserting in formula (b) the value of $E_{[x]}$ from formula (a), we have,

$$E_{[x]+1} = n_{[x]} + g_{[x]} + g_{[x]+1}$$
$$= n_{[x]} + \Sigma_0^1(g)$$

and, generally,

$$E_{[x]+t} = n_{[x]} + \Sigma_0^t(g) \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

the formula employed in Schedule (D) for deducing the rate of mortality.

It can be shown that

$$\Sigma_0^t(g) = \Sigma_0^t(g) - (bg)_{[x]+t} + (ad)_{[x]+t}$$

where g = the net movement of cases according to the Exact Duration Method; and we have, therefore, by the Nearest Duration Method (Schedule D)

$$E_{[x]+t} = n_{[x]} + \Sigma_0^t(g) - (bg)_{[x]+t} + (ad)_{[x]+t},$$

and by the Exact Duration Method (Schedule B)

$$E_{[x]+t} = n_{[x]} + \Sigma_0^t(g) - g'_{[x]+t} + (d-d')_{[x]+t},$$

(bg) and (ad) in the former formula being replaced by g' and $(d-d')$ in the latter respectively.

(b) Rate of Withdrawal.

Let s, e, represent, as before, the numbers tabulated as surviving and existing; and let

$$d_{[x]} = (ad)_{[x]} \qquad\qquad d_{[x]+t} = [(bd)_{[x]+t-1} + (ad)_{[x]+t}]$$

also let

$$f'_{[x]+t} = (e+d)_{[x]+t} + w_{[x]+t-1}$$

where $w_{[x]+t-1}$ represents the actual withdrawals in the tth year of duration; and let $g'_{x+t} = (s-f')_{[x]+t}$.

Then we have, for the numbers exposed to the risk of withdrawal

$$(wE)_{[x]} = n_{[x]} + g'_{[x]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

$$(wE)_{[x]+t} = (wE)_{[x]+t-1} + g'_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

$$= n_{[x]} + \Sigma_0^t(g') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

the formulæ employed in Schedule (D) for deducing the rate of withdrawal.

It can similarly be shown here that

$$\Sigma_0^t(g') = \Sigma_0^t(g) - (bg)_{[x]+t} + (aw)_{[x]+t}$$

and that, therefore,

$$(wE)_{[x]+t} = n_{[x]} + \Sigma_0^t(g) - (bg)_{[x]+t} + (aw)_{[x]+t}$$

which may be compared with the Exact Duration formula,

$$(wE)_{[x]+t} = n_{[x]} + \Sigma_0^t(g) - g'_{[x]+t} + (w - w')_{[x]+t}$$

(bg) in the former formula being again replaced by (g') in the latter; and (aw) by $(w - w')$.

Nearest Duration Formulæ, as Applied (in Schedules E and F) for Deducing the Rates of Mortality *and* of Withdrawal.

As the cases are sorted and tabulated by this method in a distinctive way, it will be well to investigate the formulæ independently. Let s, e, w and d, represent the numbers surviving, existing, withdrawing and dying, as tabulated by this method, so that the initial values

$$\mathrm{s}_{[x]} = (as)_{[x]} \qquad \mathrm{w}_{[x]} = (aw)_{[x]} \qquad \mathrm{d}_{[x]} = (ad)_{[x]} \qquad \mathrm{e}_{[x]} = (ae)_{[x]}$$

also

$$\mathrm{s}_{[x+t} = [(bs)_{[x]+t-1} + (as)_{[x]+t}] \qquad \mathrm{e}_{[x]+t} = [(be)_{[x]+t-1} + (ae)_{[x]+t}]$$

$$\mathrm{w}_{[x]+t} = [(bw)_{[x]+t-1} + (aw)_{[x]+t}] \qquad \mathrm{d}_{[x]+t} = [(bd)_{[x]+t-1} + (ad)_{[x]+t}]$$

and $\qquad \mathrm{f}_{[x]+t} = (\mathrm{e} + \mathrm{w} + \mathrm{d})_{[x]+t} \qquad \mathrm{g}_{[x]+t} = (\mathrm{s} - \mathrm{f})_{[x]+t}.$

Then we have

$$\bar{\mathrm{E}}_{[x]} = n_{[x]} + (as)_{[x]} - (ae)_{[x]} - (aw)_{[x]} - (ad)_{[x]} = n_{[x]} + \mathrm{g}_{[x]} \quad (8)$$

similarly,

$$\begin{aligned} \bar{\mathrm{E}}_{[x]+1} &= \bar{\mathrm{E}}_{[x]} + (bs)_{[x]} - [(be) + (bw) + (bd)]_{[x]} \\ &\qquad + (as)_{[x]+1} - [(ae) + (aw) + (ad)]_{[x]+1} \\ &= \bar{\mathrm{E}}_{[x]} + (\mathrm{s} - \mathrm{f})_{[x]+1} \\ &= \bar{\mathrm{E}}_{[x]} + \mathrm{g}_{[x]+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9) \end{aligned}$$

and generally $\qquad \bar{\mathrm{E}}_{[x]+t} = \bar{\mathrm{E}}_{[x]+t-1} + \mathrm{g}_{[x]+t} \quad . \quad . \quad . \quad . \quad (10)$

a formula for deducing the numbers exposed to risk by a continuous method.

Inserting now in formula (9) the value of $\bar{E}_{[x]}$ from formula (8), we have:

$$\bar{E}_{[x]+1}=n_{[x]}+g_{[x]}+g_{[x]+1}$$

and generally

$$\bar{E}_{[x]+t}=n_{[x]}+\Sigma_0^t(g) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

a formula for deducing the numbers exposed to risk by a process of summation, as applied in Schedule (E).

If now we analyze this formula, we have:

$$\begin{aligned}
\bar{E}_{[x]+t}&=n_{[x]}+(g_{[x]}+g_{[x]+1}+\ .\ .\ .\ .\ +g_{[x]+t}) \\
&=n_{[x]}+[(as)_{[x]}+(bs)_{[x]}+(as)_{[x]+1}+\ .\ .\ .\ .\ +(as)_{[x]+t}] \\
&\qquad -[(ae)_{[x]}+(be)_{[x]}+(ae)_{[x]+1}+\ .\ .\ .\ .\ +(ae)_{[x]+t}] \\
&\qquad -[(aw)_{[x]}+(bw)_{[x]}+(aw)_{[x]+1}+\ .\ .\ .\ .\ +(aw)_{[x]+t}] \\
&\qquad -[(ad)_{[x]}+(bd)_{[x]}+(ad)_{[x]+1}+\ .\ .\ .\ .\ +(ad)_{[x]+t}] \\
&=n_{[x]}+\Sigma_0^t(s)-(bs)_{[x]+t}-\Sigma_0^t(e)+(be)_{[x]+t} \\
&\qquad -\Sigma_0^t(w)+(bw)_{[x]+t}-\Sigma_0^t(d)+(bd)_{[x]+t} \\
&=n_{[x]}+\Sigma_0^t(g)-(bg)_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)
\end{aligned}$$

Comparing this now with formula (4) of Exact Durations,

$$\bar{E}_{[x]+t}=n_{[x]}+\Sigma_0^t(g)-g'_{[x]+t} \quad . \quad . \quad . \quad . \quad (4)$$

we see that the two formulæ differ only in their last terms, and that the difference between the numbers exposed to risk, as deduced by the Nearest and by the Exact Duration Methods, solely arises from the error introduced in the year of duration at entry (for survivors), or at exit (for cases emerging), by the assumption that $(bg)_{[x]+t}$, the balance of nearest integral durations in the year, is approximately equal to $g'_{[x]+t}$ (see pages 29, 30), the balance of true fractional exposures in the year.

APPENDIX (B).

Period of Observation Limited by Years of Duration.

Exact Duration Formulæ. (Schedule G).

Here $s_{[x]+t}$, $e_{[x]+t}$, become $s^1{}_{[x]+t}$, $e^1{}_{[x]+t}$ (the numbers surviving and existing at precise duration t), and, as the new entrants, during the period $(n_{[x]})$ now fall into rank as "survivors" at precise age $[x]$, $(s^1{}_{[x]})$, and, as there are now no cases existing at precise age $[x]$, we have $n_{[x]}=s^1{}_{[x]}$; $e^1{}_{[x]}=0$.

Representing by G and G' respectively, the modified values of g and g' under these conditions, we have

$$G_{[x]}=s^1_{[x]}-(w+d)_{[x]}, \qquad G_{[x]+t}=s^1_{[x]+t}-(e^1+w+d)_{[x]+t},$$

also, as $s'_{[x]+t}=0$, $e'_{[x]+t}=0$, we have

$$G'_{[x]+t}=-(w'+d')_{[x]+t}$$

for all values of t.

Then formula (1) becomes

$$\bar{\mathrm{E}}_{[x]}=G_{[x]}-G'_{[x]}=G_x+(w'+d')_{[x]} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Similarly, formula (3) becomes

$$\begin{aligned}\bar{\mathrm{E}}_{[x]+t}&=\bar{\mathrm{E}}_{[x]+t-1}+G_{[x]+t}-\Delta G'_{[x]+t-1}\\&=\bar{\mathrm{E}}_{[x]+t-1}+G_{[x]+t}+\Delta(w'+d\,)_{[x]+t-1} \quad . \quad (14)\end{aligned}$$

also formula (4) becomes

$$\begin{aligned}\bar{\mathrm{E}}_{[x]+t}&=\Sigma_0^t(G)-G'_{[x]+t}\\&=\Sigma_0^t(G)+(w'+d')_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)\end{aligned}$$

These formulæ give the numbers exposed to risk ($\bar{\mathrm{E}}$), from which the usual functions (E) and (wE) can be deduced. It is however, in this case equally convenient to deduce these latter functions by a direct process. For we have

$$\bar{\mathrm{E}}=\mathrm{E}+(d'-d)$$

and $$\bar{\mathrm{E}}=(wE)+(w'-w),$$

and inserting these values successively in formulæ (13), (14) and (15) we have for the numbers exposed to the risk of death,

$$\mathrm{E}_{[x]}=G_{[x]}+(w'+d)_{[x]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\mathrm{E}_{[x]+t}=\mathrm{E}_{[x]+t-1}+G_{[x]+t}+\Delta(w'+d)_{[x]+t-1} \quad . \quad . \quad (17)$$

and $$\mathrm{E}_{[x]+t}=\Sigma_0^t(G)+(w'+d)_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Similarly, for the numbers exposed to the risk of withdrawal, we have

$$(wE)_{[x]}=G_{[x]}+(w+d')_{[x]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$(wE)_{[x]+t}=(wE)_{[x]+t-1}+G_{[x]+t}+\Delta(w+d')_{[x]+t-1} \quad (20)$$

$$(wE)_{[x]+t}=\Sigma_0^t(G)+(w+d')_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Mean Duration Formulæ. (Schedule H.)

Here $w'=\dfrac{w}{2}$, $d'=\dfrac{d}{2}$, and formulæ (16), (17) and (18) become respectively

$$E_{[x]} = G_{[x]} + \left(\frac{w}{2} + d\right)_{[x]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$E_{[x]+t} = E_{[x]+t-1} + G_{[x]+t} + \Delta\left(\frac{w}{2} + d\right)_{[x]+t-1} \quad . \quad . \quad (23)$$

$$E_{[x]+t} = \Sigma_0^t(G) + \left(\frac{w}{2} + d\right)_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Similarly, for the numbers exposed to the risk of withdrawal, formulæ (19), (20) and (21) become respectively

$$(wE)_{[x]} = G_{[x]} + \left(w + \frac{d}{2}\right)_{[x]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$(wE)_{[x]+t} = (wE)_{[x]+t-1} + G_{[x]+t} + \Delta\left(w + \frac{d}{2}\right)_{[x]+t-1} \quad (26)$$

$$(wE)_{[x]+t} = \Sigma_0^t(G) + \left(w + \frac{d}{2}\right)_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Nearest Duration Formulæ. (Schedules J, K.)

Reverting to, and analyzing formula (11) we have

$$\bar{E}_{[x]+t} = n_{[x]} + \Sigma_0^t(\mathrm{g})$$
$$= n_{[x]} + \Sigma_0^t[\mathrm{s} - (\mathrm{e} + \mathrm{w} + \mathrm{d})]$$

which becomes, when the observation is limited by years of duration (since $\mathrm{s} = s^1$, $\mathrm{e} = e^1$, and $n_{[x]} = s^1{}_{[x]}$),

$$\bar{E}_{[x]+t} = \Sigma_0^t[s^1 - (e^1 + \mathrm{w} + \mathrm{d})],$$

which we may call $\Sigma_0^t(\mathrm{G})$; therefore

$$\bar{E}_{[x]+t} = \Sigma_0^t(\mathrm{G}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

but, as $\bar{E}_{[x]+t} = E_{[x]+t} - (ad)_{[x]+t}$

$$= (wE)_{[x]+t} - (aw)_{[x]+t}$$

we obtain the following direct formulæ for the numbers exposed to the risk of death and of withdrawal respectively

$$E_{[x]+t} = \Sigma_0^t(\mathrm{G}) + (ad)_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and $\quad (w\mathrm{E})_{[x]+t} = \Sigma_0^t(\mathrm{G}) + (aw)_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$

Here it may be shown that

$$\Sigma_0^t(\mathrm{G}) = \Sigma_0^t(G) + [(bw) + (bd)]_{[x]+t},$$

and the above formulæ thus become

$$E_{[x]+t} = \Sigma_0^t(G) + [(bw) + d]_{[x]+t} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and $\quad (wE)_{[x]+t} = \Sigma_0^t(G) + [w + (bd)]_{[x]+t} \quad . \quad . \quad . \quad . \quad (32)$

which may be compared with the Exact Duration formulæ (18) and (21), the quantities w' and d' being replaced, in formulæ (31) and (32), by (bw) and (bd), respectively.

Appendix (C).

The values of $E_{[x]+t}$ and of $(wE)_{[x]+t}$, as deduced from that of $\overline{E}_{[x]+t}$ by the several formulæ (see pages 5, 15, and 18)

$$E_{[x]+t}=\overline{E}_{[x]+t}+(d-d')_{[x]+t} \qquad (wE)_{[x]+t}=\overline{E}_{[x]+t}+(w-w')_{[x]+t}$$

$$E_{[x]+t}=\overline{E}_{[x]+t}+\frac{d_{[x]+t}}{2} \qquad (wE)_{[x]+t}=\overline{E}_{[x]+t}+\frac{w_{[x]+t}}{2}$$

$$E_{[x]+t}=\overline{E}_{[x]+t}+(ad)_{[x]+t} \qquad (wE)_{[x]+t}=\overline{E}_{[x]+t}+(aw)_{[x]+t}$$

according to the Exact, Mean, and Nearest Duration Methods respectively, do not appear, in the particular case here investigated, to represent quite correctly the true relations. As the period of observation terminates at the close of a calendar year, the cases "existing" are necessarily under observation for a portion only of the year of duration then current, and some of the cases of death (and of withdrawal) during the last calendar year, would, if treated as "existing", have in like manner completed only a portion of the year of duration current at exit. In strictness, therefore, such cases should contribute to the number exposed to risk, not the full year of duration current at exit, but only that portion of the year which actually fell within the period of observation.

The effect of the assumptions made, and the nature and extent of the correction required, will be seen if we suppose that 1,000 members complete their year of duration in the last calendar year of the period of observation, on 30 June; and that of these 1,000 cases none withdraw, but five die during the following six months. The number exposed to risk in respect of the 995 survivors would clearly be $=\frac{995}{2}=497{\cdot}5$; and to this would be added, by the above formulæ, a full year's exposure in respect of each of the five deaths, making the total number exposed to risk 502·5. The annual rate of mortality deduced in respect of these cases only would thus be $\frac{5}{502{\cdot}5}={\cdot}00995$; but it would seem to be clear that the true annual rate of mortality $={\cdot}01$, and that the number exposed to risk should, in strictness, be $497{\cdot}5+2{\cdot}5=500$; that is, that the death cases should be considered as exposed to risk for six months each only.

The cases affected are those of death (or withdrawal) occurring in the last calendar year of the period of observation, where the anniversary of entry in that year precedes the date of death (or withdrawal). Assuming that this would include about one-half of the number of cases emerging in the year; that the number of emergents in each year of a quinquennial period do not greatly vary; and that, upon the average, two-thirds of the year of duration current at exit falls, in these cases, within the period of observation; the amount by which the exposures of the cases of death (or of withdrawal) would be overstated, may be roughly estimated at about one-thirtieth part (or $3\frac{1}{3}$ per-cent) of the total number of deaths (or withdrawals) occurring in the quinquennium.

I find that, of the 481 withdrawals included in the illustrative experience employed in this paper, 34 occurred in the calendar year 1892, subsequently to their anniversaries of entry in that year; and that, of these 34 cases, the actual portion of the year of duration current at exit which fell within the period of observation, was, in the aggregate 25·5 years. Of the 48 deaths, seven occurred in the calendar year 1892, subsequently to their anniversaries of entry in that year; in respect of which the actual portion of the current year of duration which fell within the period was 5·7 years. The number exposed to the risk of withdrawals was thus overstated by (34—25·5=) 8·5 years; and the number exposed to the risk of death, by (7—5·7=) 1·3 years. The aggregate years of risk being 6,660 and 6,458 respectively, the amount of the error was (in this particular case) about 13 per 10,000 in the numbers exposed to the risk of withdrawal, and about two per 10,000 in the numbers exposed to the risk of death. The correction is, therefore, extremely small, and may, in practice, be disregarded.

Appendix (D) (see page 45).

In the case of an experience where the value of d_x tends to increase steadily with the age, the probability will be for the death cases to congregate towards the end of the year of duration, if we assume that the progression of the decrements in *half-years* follows generally the same law as their annual progression. In the H^M Table, for example, the value of d increases between ages 14 and 22, and between ages 25 and 74, after which it steadily diminishes until the end of the table; and with such an experience, it would appear to be more probable that death will happen (on the whole) in the later half of the year of duration. Under the H^F Table the same tendency is observable, but in a less strongly marked form; the value of d increasing between ages 10 to 16, 19 to 28, 39 to 45, 49 to 76, after which it steadily diminishes.

Appendix (E).

The formulæ given on pp. 72, 73, for the values of q' and of q'', are closely approximate, the true formulæ being, upon the assumption of a uniform distribution of deaths throughout the year,

$$q'=\frac{q-\dfrac{q\cdot(wq)}{2}}{1-\dfrac{q\cdot(wq)}{4}} \qquad q''=\frac{q+(wq)-q\cdot(wq)}{1-\dfrac{q\cdot(wq)}{4}}$$

These formulæ are deduced directly from those given by Dr. Sprague (*J.I.A.*, xxi, 416–7) for computing the elements of the mortality table in the case of a double decrement, which, expressed in the notation here employed, become

$$d=l.\frac{q\left(1-\dfrac{(wq)}{2}\right)}{1-\dfrac{q\cdot(wq)}{4}}=l.\frac{m}{1+\dfrac{m+(wm)}{2}}$$

$$w = l\,\frac{(wq)\left(1-\frac{q}{2}\right)}{1-\frac{q.(wq)}{4}} = l.\frac{(wm)}{1+\frac{m+(wm)}{2}}$$

$$l_{+1} = l\left[1-\frac{q+(wq)-q(wq)}{1-\frac{q.(wq)}{4}}\right] = l.\frac{1-\frac{m+(wm)}{2}}{1+\frac{m+(wm)}{2}}$$

The alternative expressions given on the right-hand side are convenient for deducing the mortality elements in terms of the *central rates* of death and of withdrawal; and, in the case of an experience such as that here investigated, it is, on the whole, more convenient to compute the values of $m_{[x]+t}$ and of $(wm)_{[x]+t}$ direct from that of $\bar{E}_{[x]+t}$, and thence to deduce the values of $l_{[x]+t}$, $d_{[x]+t}$, and $w_{[x]+t}$. The values of $E_{[x]+t}$ and of $(wE)_{[x]+t}$, as well as those of $q_{[x]+t}$, and of $(wq)_{[x]+t}$, can then be dispensed with.

It will be convenient to call

$$\mathrm{m}_{[x]+t} = \frac{m}{1+\frac{m+(wm)}{2}} \qquad (\mathrm{wm})_{[x]+t} = \frac{(wm)}{1+\frac{m+(wm)}{2}}$$

and

$$1-[\mathrm{m}_{[x]+t}+(\mathrm{wm})_{[x]+t}] = \frac{1-\frac{m+(wm)}{2}}{1+\frac{m+(wm)}{2}}$$

then we have,

$$d_{[x]+t} = l_{[x]+t}\,.\,\mathrm{m}_{[x]+t} \qquad w_{[x]+t} = l_{[x]+t}(\mathrm{wm})_{[x]+t}$$

and

$$l_{[x]+t+1} = l_{[x]+t}[1-(\mathrm{m}_{[x]+t}+(\mathrm{wm})_{[x]+t})]$$

and by these formulæ the values of d, w, and l, can be readily computed.

The appended Table IX shows, for age at entry (20), the central rates of mortality and of withdrawal, the logarithms of the factors m, (wm), and $\{1-[\mathrm{m}+(\mathrm{wm})]\}$, respectively, and the numbers living, dying, and withdrawing, as deduced in each successive year of duration by the above formulæ. The rate of withdrawal here employed is a graduated rate, based upon that experienced in years of duration, as set out in column (8) of Table (II).

In Table X the Commutation columns are deduced, at a rate of interest of 3 per-cent per annum, from the mortality elements given in Table IX; also the values of assurances and annuities, and of the benefit during non-employment; with allowance, throughout, for mortality and withdrawal.

Table IX.—Mortality Table.

Table showing the Central Rates of Mortality and of Withdrawal, and the Mortality Table (with decrements by Death and Withdrawal) as deduced therefrom.—Central Age at Entry $[x]=20$.

Duration	Central Rate of Mortality	Central Rate of Withdrawal	Logarithms of Factors for Deducing Mortality Table			Mortality Table: Numbers Living	Deaths	Withdrawals	Deaths and Withdrawals
(t)	$m_{[x]+t}$	$(wm)_{[x]+t}$	log $m_{[x]+t}$	log $(wm)_{[x]+t}$	log $[1-(m+wm)]_{[x]+t}$	$l_{[x]+t}$	$d_{[x]+t}$	$w_{[x]+t}$	$(d+w)_{[x]+t}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	·00461	·16220	7·62891	9·17526	9·92739	100,000	426	14,970	15,396
1	·00471	·13330	·64404	·09585	·93997	84,604	373	10,549	10,922
2	·00481	·11080	·65775	·02014	·94974	73,682	335	7,717	8,052
3	·00491	·09424	·67007	8·95323	·95691	65,630	307	5,892	6,199
4	·00521	·08333	·69803	·90199	·96153	59,431	297	4,741	5,038
5	·00541	·07792	·71547	·87392	·96379	54,393	282	4,070	4,352
6	·00572	·06718	·74185	·81169	·96832	50,041	276	3,244	3,520
7	·00592	·05656	·75896	·73915	·97286	46,521	267	2,551	2,818
8	·00622	·05128	·78148	·69764	·97502	43,703	264	2,179	2,443
9	·00652	·04604	·80298	·65187	·97716	41,260	262	1,852	2,114
10	·00682	·04604	·82245	·65181	·97704	39,146	260	1,756	2,016
11	·00713	·04604	·84170	·65175	·97691	37,130	258	1,664	1,922
12	·00733	·04604	·85366	·65170	·97681	35,208	251	1,580	1,831
13	·00763	·04082	·87214	·60048	·97896	33,377	249	1,329	1,578
14	·00793	·04082	·88881	·60041	·97882	31,799	246	1,268	1,514
15	·00823	·03562	·90598	·54227	·98095	30,285	244	1,056	1,300
16	·00854	·03046	·92307	·47534	·98306	28,985	243	865	1,108
17	·00884	·02532	·93909	·39610	·98516	27,877	242	695	937
18	·00924	·02532	·95823	·39602	·98499	26,940	245	670	915
19	·00965	·02020	·97809	·29891	·98703	26,025	248	518	766
20	·01015	·02020	·99993	·29881	·98682	25,259	253	502	755
21	·01066	·02020	8·02111	·29870	·98660	24,504	257	487	744
22	·01126	·02020	·04476	·29857	·98633	23,760	263	474	737
23	·01197	·01511	·07225	·17342	·98824	23,023	272	343	615
24	·01268	·01511	·09712	·17326	·98792	22,408	280	334	614
25	·01339	·01511	·12064	·17312	·98763	21,794	288	324	612
26	·01430	·01511	·14900	·17292	·98723	21,182	298	316	614
27	·01511	·01005	·17383	7·99674	·98907	20,568	307	204	511
28	·01603	·01005	·19930	·99654	·98867	20,057	317	200	517
29	·01684	·00501	·22162	·69512	·99051	19,540	325	97	422
30	·01776	...	·24560	...	·99229	19,118	337	...	...
31	·01877	...	·26940	...	·99185	18,781	349	...	...
32	·01990	...	·29455	...	·99136	18,432	363	...	...
33	·02132	...	·32419	...	·99075	18,069	381	...	...
34	·02327	...	·36177	...	·98989	17,688	407	...	...
35	·02583	...	·40655	...	·98879	17,281	440	...	...
36	·02891	...	·45482	...	·98744	16,841	480	...	...
37	·03262	...	·50645	...	·98583	16,361	525	...	...
38	·03728	...	·56346	...	·98381	15,836	580	...	...
39	·04102	...	·60418	...	·98218	15,256	613	...	...

Table X.—Values of Benefits.

Table showing the Commutation Columns, allowing for Mortality and Withdrawal, and the values of Annuities, Assurances and Allowance during Non-Employment, as deduced therefrom.—Central Age at Entry $[x]=20$.—*Interest at* 3 *per-cent per annum.*

Duration	Commutation Columns, Allowing for Mortality and Withdrawal				Values of Annuity and Assurance		Rate of Non-Employment	Commutation Columns, Allowing for Mortality and Withdrawal		Values of Allowance
(t)	$D''_{[x]+t}$	$N''_{[x]+t}$	$C'_{[x]+t}$	$M'_{[x]+t}$	$a''_{[x]+t}$	$A'_{[x]+t}$	$u_{[x]+t}$	$D''_{[x]+t+\frac{1}{2}} \cdot u_{[x]+t}$	$\Sigma(D''u)$	$U_{[x]+t}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0	55,368·	488,939·	228·7	5,721·0	8·831	·104	·252	6,353·0	146,845·	2·652
1	45,479·	443,460·	194·5	5,492·3	9·751	·121	·250	10,492·0	140,442·	3·088
2	38,454·	405,006·	169·8	5,297·8	10·532	·138	·249	8,927·7	129,949·	3·379
3	33,254·	371,752·	151·0	5,128·0	11·170	·155	·249	7,780·0	121,022·	3·640
4	29,236·	342,516·	141·6	4,977·0	11·716	·170	·249	6,874·2	113,242·	3·873
5	25,978·	316,538·	131·0	4,835·4	12·185	·187	·248	6,098·6	106,368·	4·095
6	23,204·	293,334·	124·3	4,704·4	12·642	·203	·246	5,430·2	100,269·	4·321
7	20,944·	272,390·	116·7	4,580·1	13·006	·219	·242	4,845·6	94,839·	4·528
8	19,102·	253,288·	112·1	4,463·4	13·260	·234	·237	4,338·5	89,993·	4·711
9	17,509·	235,779·	108·0	4,351·3	13·467	·249	·231	3,885·2	85,655·	4·892
10	16,128·	219,651·	104·0	4,243·3	13·620	·263	·226	3,500·7	81,770·	5·070
11	14,852·	204,799·	100·1	4,139·3	13·790	·279	·219	3,123·6	78,269·	5·270
12	13,673·	191,126·	94·8	4,039·2	13·979	·295	·214	2,809·6	75,145·	5·496
13	12,584·	178,542·	91·0	3,944·4	14·188	·313	·211	2,555·7	72,336·	5·748
14	11,640·	166,902·	87·5	3,853·4	14·339	·331	·210	2,352·4	69,780·	5·995
15	10,763·	156,139·	84·1	3,765·9	14·507	·350	·213	2,211·6	67,428·	6·265
16	10,001·	146,138·	81·3	3,681·8	14·613	·368	·220	2,127·3	65,216·	6·521
17	9,338·1	136,800·	78·8	3,600·5	14·650	·386	·229	2,072·4	63,089·	6·756
18	8,761·7	128,038·	77·3	3,521·7	14·614	·402	·238	2,020·6	61,016·	6·964
19	8,217·5	119,821·	75·8	3,444·4	14·581	·419	·244	1,947·2	58,996·	7·179
20	7,743·4	112,078·	75·2	3,368·6	14·474	·435	·249	1,872·0	57,048·	7·369
21	7,293·1	104,785·	74·3	3,293·4	14·367	·452	·250	1,769·9	55,176·	7·566
22	6,865·6	97,919·	73·9	3,219·1	14·262	·469	·252	1,666·3	53,406·	7·743
23	6,459·1	91,460·	74·1	3,145·2	14·160	·487	·256	1,608·0	51,740·	8·010
24	6,103·4	85,356·	74·1	3,071·1	13·985	·503	·261	1,548·6	50,132·	8·214
25	5,763·0	79,593·	73·9	2,997·0	13·811	·520	·268	1,501·0	48,584·	8·430
26	5,438·1	74,155·	74·4	2,923·1	13·635	·537	·276	1,457·9	47,083·	8·658
27	5,126·7	69,029·	74·3	2,848·7	13·465	·556	·284	1,417·2	45,625·	8·899
28	4,853·7	64,175·	74·6	2,774·4	13·222	·572	·293	1,384·5	44,207·	9·108
29	4,591·0	59,584·	74·2	2,699·8	12·979	·588	·303	1,356·2	42,823·	9·328
30	4,360·9	55,223·	74·5	2,625·6	12·663	·602	·316	1,346·2	41,467·	9·509
31	4,159·4	51,064·	75·1	2,551·1	12·277	·613	·335	1,360·6	40,121·	9·646
32	3,963·2	47,100·	75·8	2,476·0	11·884	·625	·358	1,384·6	38,760·	9·780
33	3,772·0	43,328·	77·3	2,400·2	11·487	·636	·384	1,412·5	37,375·	9·909
34	3,584·9	39,743·	80·1	2,322·9	11·086	·648	·416	1,453·0	35,963·	10·032
35	3,400·4	36,343·	84·2	2,242·8	10·688	·660	·454	1,502·2	34,510·	10·150
36	3,217·3	33,126·	89·0	2,158·6	10·296	·671	·499	1,559·8	33,008·	10·260
37	3,034·5	30,091·	94·6	2,069·6	9·916	·682	·538	1,583·3	31,441·	10·363
38	2,851·5	27,240·	101·3	1,975·0	9·553	·693	·577	1,592·2	29,865·	10·473
39	2,667·2	24,573·	104·1	1,873·7	9·213	·702	·616	1,587·0	28,272·	10·600